Quantum Error Correction During 50 Gates

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Fault tolerant protocol assumes the application of error correction after every quantum gate. However, correcting errors is costly in terms of time and number of qubits. Here we demonstrate that quantum error correction can be applied significantly less often with only a minimal loss of fidelity. This is done by simulating the implementation of 50 encoded, single-qubit, quantum gates within the [[7,1,3]] quantum error correction code in a noisy, non-equiprobable Pauli error environment with error correction being applied at different intervals. We find that applying error correction after every gate is rarely optimal and even applying error correction only once after all 50 gates, though not generally optimal, sacrifices only a slight amount of fidelity with the benefit of 50-fold saving of resources. In addition, we find that in cases where bit-flip errors are dominant, it is best not to apply error correction at all.

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Standard approaches to quantum fault tolerance (QFT), the computational framework that allows for successful quantum computation despite a finite probability of error in basic computational gates [4–7], assume the application of quantum error correction (QEC) [1–3] after every operation. QEC codes protect quantum information by storing some number of logical qubits in a subspace of a greater number of physical qubits thus forming the building blocks for QFT. However, the syndrome measurements needed to check for and correct errors are very expensive in terms of number of qubits required and implementation time. In this paper we demonstrate via numerical simulations that applying QEC after every operation is not necessary and, in general, not optimal. The simulations are done for single-logical-qubit operations on information encoded in the [[7,1,3]] QEC code [8].

A guiding principle of QFT is to implement all protocols in such a way so as to ensure that information does not leave the encoded subspace (and become subject to errors). Only specialized gates can adhere to this principle. Nevertheless, for many QEC codes universal quantum computation can be performed within the QFT framework if the gate set is restricted to Clifford gates plus the T-gate, a single-qubit $\pi/4$ phase rotation. It is not *a priori* obvious how to implement general gates using such a restricted gate set. A method for implementing an arbitrary single-qubit rotation (within prescribed accuracy ϵ) within these constraints was initially explored in [9, 10] and has recently become an area of intense investigation [11–17]. For Calderbank-Shor-Steane (CSS) codes, Clifford gates can be implemented bit-wise while the T-gates require a specially prepared ancilla state and a series of controlled-NOT gates. Thus, the primary goal of these investigations has been to construct circuits within ϵ of a desired (arbitrary) rotation while limiting the number of resource-heavy T-gates. As an example, a σ_z rotation by .1 can be implemented with accuracy better than 10^{-5} using 56 [17] T-gates, interspersed by at least as many single-qubit Clifford gates. QFT would suggest that QEC be applied after each one of the more than 100 gates needed to implement such a rotation requiring thousands of additional qubits and hundreds of time steps. Adhering to this is thus very resource intensive.

Recently there have been a number of attempts to reduce the resource consumption of a quantum computation by carefully analyzing, simulating, and comparing protocols within the QFT framework [18–22]. Specifically, it was shown that QEC need not be applied after every gate and, in fact, should not be applied after every gate [23]. Applying QEC less often will consume less resources, while still enabling successful quantum computation. This point was also made, though addressed in a different way, in Ref. [24]. Here we numerically simulate the implementation of 50 logical gates on information encoded into the [[7,1,3]] QEC code applying QEC (via syndrome measurements and possible recovery operations) at different intervals and determining which scheme is best for different error probabilities. The simulations are explicit, the entire density matrix is calculated at every step.

The [[7,1,3]] or Steane QEC code will correct an error on one physical qubit of a seven qubit system that encodes one qubit of quantum information. If errors occur on two (physical) qubits the code will be unable to restore the system to its proper state. By applying gates following the rules of QFT, we can ensure that the probability of an error occurring on two physical qubits remains of order p^2, where p is the probability of a single qubit error per gate, no matter how many gates are applied. Thus, if p is small enough one need only apply QEC at the end of the sequence. However, for long sequences of gates it is likely that p is not that small and the coefficients in front of the higher order error terms will grow to an unacceptable level. QEC would then be needed more often.

Of course, if QEC could be implemented perfectly, and we have unlimited resources available, it would be worthwhile to apply QEC as much as possible. In reality, QEC cannot be done perfectly and we are extremely concerned about resource consumption. Thus, we are left to ask, how often should QEC be applied?

To address this we simulate 50 single-qubit gates on the [[7,1,3]] QEC code in a nonequiprobable Pauli operator error environment [25] with non-correlated errors. As in [26], this error model is a stochastic version of a biased noise model that can be formulated in terms of Hamiltonians coupling the system to an environment. Here, different error types arise with different arbitrary probabilities. Individual qubits undergo σ_x^j errors with probability p_x, σ_y^j errors with probability p_y, and σ_z^j errors with probability p_z, where σ_i^j, $i = x, y, z$ are the Pauli spin operators on qubit j. We assume that only qubits taking part in a gate operation, initialization, or measurement will be subject to error while other qubits are perfectly stored. This idealized assumption is partially justified in that idle qubits may be less likely to undergo error than those involved in gates (see for example [27]).

We assume a single qubit state $|\psi\rangle = \cos\alpha|0\rangle + e^{i\beta}\sin\alpha|1\rangle$, perfectly encoded into the [[7,1,3]] error correction code. We then implement a series of gates, $U_{50}...U_2U_1$, in the nonequiprobable error environment leading to a final state, ρ_f, of the 7 qubits. To determine the accuracy of the simulated implementations with perfectly applied gates, ρ_i, we utilize the state fidelity $F(\rho_i, \rho_f) = \mathrm{Tr}[\rho_i\rho_f]$. In addition we will find it useful to utilize the infidelity $I(\rho_i, \rho_f) = 1 - F(\rho_i, \rho_f)$, and a logarithmic infidelity $log_{10}[I(\rho_i, \rho_f)]$.

Our choice of gates stems from the above noted work on the implementation of arbitrary single qubit gates with gates from the set Clifford plus T. We define the composite gates $A = HPT$ and $B = HT$ and simulate the implementation of the 50 gates:

$$U = ABBBAAAABBABABABBBAA. \qquad (1)$$

We then formulate 7 different error correction application schemes: applying QEC after every gate (50 QEC applications), after every composite gate A and B (20 applications), after every other composite gate (10 applications), after every 5 composite gates (4 applications), after each half of the sequence U (2 applications), only after the entire sequence (1 application), and not at all. Each scheme is simulated for 64 error models: each p_j, $j = x, y, z$, takes all values $10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}$. For the initial state we use the basis state $|0\rangle$. Other tested initial states and gate sequences give similar results.

Implementing a Clifford gate, C, on the [[7,1,3]] QEC code requires implementing $C^\dagger$ on each of the 7 qubits. To implement a logical T-gate on a state encoded in the [[7,1,3]] QEC code requires constructing the ancilla state $|\Theta\rangle = \frac{1}{\sqrt{2}}(|0_L\rangle + e^{i\frac{\pi}{4}}|1_L\rangle)$, where $|0_L\rangle$ and $|1_L\rangle$ are the logical basis states on the [[7,1,3]] QEC code. Bit-wise CNOT gates are then applied between the state $|\Theta\rangle$ and the encoded state with the $|\Theta\rangle$ state qubits as control. Measurement of zero on the encoded state projects the encoded state with the application of a T-gate onto the qubits that had made up the $|\Theta\rangle$ state. Our simulations are done in a fault tolerant fashion following [20].

Results from the simulations are depicted in Fig. 2 and Tables I and II. Fig. 2 shows the logarithmic infidelity for the output state upon the application of 50 gates to the initial state $|0\rangle$ with QEC applied after each gate (left) and not at all (right). When no error correction is applied the fidelity appears to decrease steadily with error probability independent of type of error. When error correction is applied not all errors are equally damaging. Bit-flip, σ_x errors are clearly seen to be more harmful to the fidelity than either σ_y or σ_z errors. Ref. [29] demonstrates that this is due to the choice of applying the bit flip syndromes before the phase flip syndromes and due to the noisy construction of the Shor states.

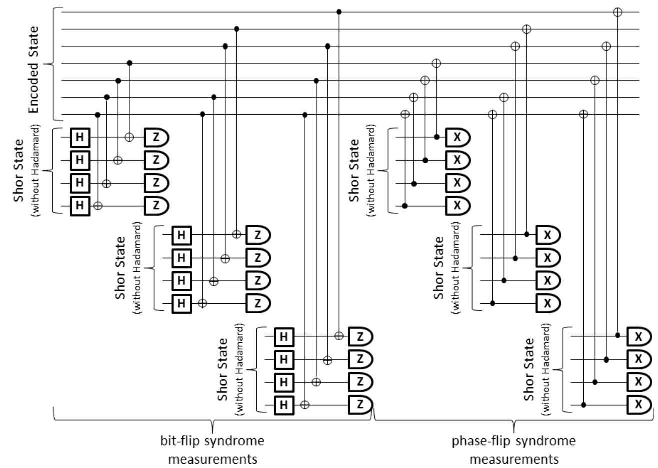

FIG. 1: Circuit for syndrome measurements for the [[7,1,3]] QEC code. Syndrome measurement is done in a fault tolerant fashion using four-qubit Shor states, GHZ states with a Hadamard applied to each qubit. The Shor states themselves are constructed in the nonequiprobable Pauli operator error environment. In addition, each syndrome is repeated twice.

The other QEC application schemes provide only slight advantages or disadvantages to the case of QEC after every gate. The resolution of the above plots are not fine enough to demonstrate these differences which are instead collected in the Tables below. These tables show which QEC application scheme is optimal with respect to fidelity and also highlight how small the difference in fidelity is whether applying QEC 50 times or only once.

The two tables show results for each QEC application scheme for different error probabilities: Table I is for depolarization, $p = p_x = p_y = p_z$, while Table II is for error models where one error probability, p_i, changes while the other two remain constant $p_j = p_k = 10^{-10}$. In each Table the top line gives the infidelity for the scheme where QEC is applied after every gate (50 times), I_{50}. Lower lines show the fractional change, D, in the infidelity upon using other QEC application schemes where:

$$D(I_{50}, I_q) = \frac{I_{50} - I_q}{I_{50}} \qquad (2)$$

and $q = 20, 10, 4, 2, 1, 0$. Note that a positive fractional change means that the infidelity is lower when using less QEC and thus the fidelity is higher. In other words, positive fractional change means a higher fidelity when

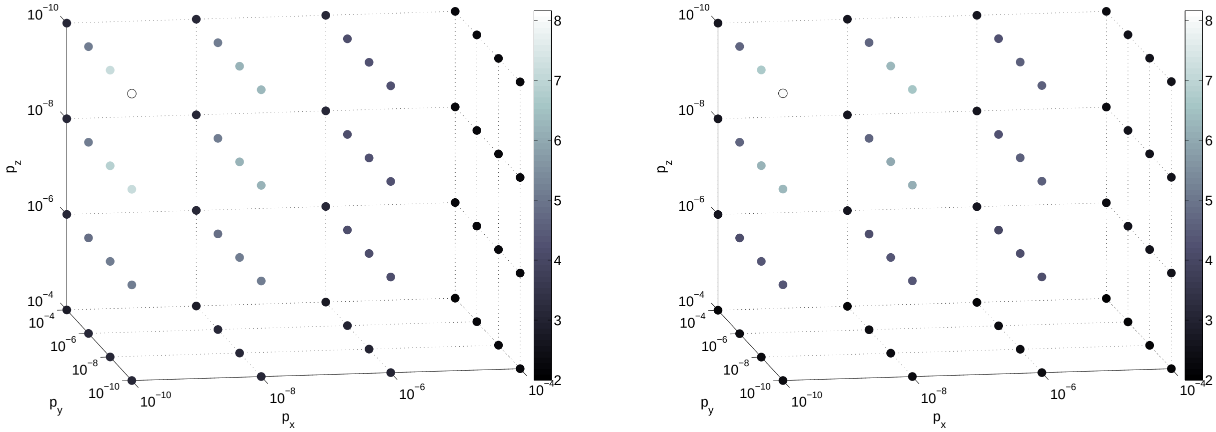

FIG. 2: Logarithmic infidelity of ρ_f for 50 gates applied to initial state $|0\rangle$ with (left) error correction applied after every gate and (right) no QEC at all. On this color scale, the results of other QEC application schemes would be indistinguishable from the results of QEC applied after every gate. We note that in the case of no QEC applications the decrease in fidelity is about equal for each of the three error probabilities. However, when QEC is applied after every gate the σ_x errors clearly lead to lower fidelities than the other errors. In fact when σ_x errors are dominant the fidelities are even lower than when not applying QEC at all.

TABLE I: Second line: infidelity of final state after 50 noisy gates with noisy QEC applied after each as a function of depolarization strength $p = p_x = p_y = p_z$. Lower lines: fractional increase or decrease of infidelity for different QEC application schemes compared to the case of QEC after every gate.

QEC applications	$p = 10^{-10}$	$p = 10^{-8}$	$p = 10^{-6}$	$p = 10^{-4}$
50	6.90×10^{-9}	6.90×10^{-7}	6.90×10^{-5}	7.29×10^{-3}
20	-1.61×10^{-8}	1.84×10^{-7}	1.82×10^{-5}	1.88×10^{-3}
10	-1.11×10^{-6}	1.26×10^{-7}	1.28×10^{-5}	1.38×10^{-3}
4	-5.63×10^{-7}	7.37×10^{-8}	7.55×10^{-6}	8.84×10^{-4}
2	-8.37×10^{-7}	6.56×10^{-8}	6.69×10^{-6}	8.05×10^{-4}
1	-1.48×10^{-6}	5.87×10^{-8}	5.95×10^{-6}	7.35×10^{-4}
0	$-.391$	$-.391$	$-.391$	$-.372$

using less QEC. Negative fractional change means the fidelity is highest when applying QEC after every gate. We should quickly note, however, that even if applying QEC after every gate gives the highest fidelity, this does not mean it is the optimal choice of QEC application scheme. If the fractional change, $D(I_{50}, I_1)$ is small one may achieve an almost optimal fidelity while saving a factor of up to 50 in time and number of qubits, perhaps a worthwhile tradeoff.

Returning to Table I we note that applying QEC after every gate gives the optimal fidelity for the lowest value of p. This is not surprising as we would expect that as error probabilities approach zero the QEC implementations become less and less error prone and the penalty in fidelity for the implementations essentially vanishes. Consistent with this is that as the number of QEC applications increase the lower the fidelity such that when there is only one application the fractional change is 1.5×10^{-6}. We note that even this greatest decrease of fractional change translates to a decrease of fidelity of only 1.0×10^{-14}. Such a small change may not warrant 50 times the time and number of qubits.

When the depolarization strength is increased the application of QEC after every gate becomes non-optimal. In fact, it is the worst of all the QEC application schemes except for not applying QEC at all. The best scheme is $q = 20$, when QEC is applied after every composite gate A or B, and the fidelity worsens as as the number of QEC applications decrease. Why should the fidelity increase with more error correction except when it is applied at every gate? Previous work [23] showed explicitly that for up to four gates one of which is a T-gate there was no need to apply QEC more often than once at the end of the sequence. This is exactly what is done in the $q = 20$ case. Apparently, for depolarization, implementing multiple T-gates before error correction allows for too many possible errors and hence a lower fidelity.

When the error model is asymmetric we see widely varying results depending on the degree of asymmetry and which errors are dominant as demonstrated in Table II. When σ_y errors are dominant applying QEC after each gate is always optimal and, in general, more QEC applications lead to lower fidelites. When σ_z errors are dominant, applying QEC after every gate is the worst

TABLE II: Second line: one minus the fidelity of final state after 50 noisy gates with noisy QEC applied after each gate as a function of p_i with $p_j = p_k = 10^{-10}$. Lower lines: percent increase or decrease of one minus the fidelity of different QEC application schemes compared to applying QEC after every gate as a function of p_i.

q	$p_x = 10^{-8}$	$p_x = 10^{-6}$	$p_x = 10^{-4}$	$p_y = 10^{-8}$	$p_y = 10^{-6}$	$p_y = 10^{-4}$	$p_z = 10^{-8}$	$p_z = 10^{-6}$	$p_z = 10^{-4}$
50	5.5×10^{-7}	5.5×10^{-5}	5.5×10^{-3}	7.6×10^{-8}	7.0×10^{-6}	7.4×10^{-4}	7.6×10^{-8}	7.0×10^{-6}	8.3×10^{-4}
20	1.5×10^{-8}	3.9×10^{-9}	3.1×10^{-5}	-1.7×10^{-7}	-1.7×10^{-6}	-1.6×10^{-4}	1.2×10^{-6}	1.3×10^{-4}	1.1×10^{-2}
10	1.4×10^{-8}	-8.7×10^{-8}	3.0×10^{-5}	-1.5×10^{-7}	-1.7×10^{-6}	-1.7×10^{-4}	1.1×10^{-6}	1.3×10^{-4}	1.1×10^{-2}
4	-1.8×10^{-9}	-8.6×10^{-8}	3.8×10^{-5}	-3.0×10^{-7}	-1.7×10^{-6}	-1.7×10^{-4}	1.1×10^{-6}	1.3×10^{-4}	1.1×10^{-2}
2	3.4×10^{-9}	-8.6×10^{-8}	3.9×10^{-5}	-2.5×10^{-7}	-1.7×10^{-6}	-1.7×10^{-4}	1.1×10^{-6}	1.3×10^{-4}	1.1×10^{-2}
1	3.0×10^{-9}	-8.6×10^{-8}	4.0×10^{-5}	-2.9×10^{-7}	-1.7×10^{-6}	-1.7×10^{-4}	1.1×10^{-6}	1.3×10^{-4}	1.1×10^{-2}
0	.479	.491	.488	−1.85	−2.00	−1.89	−5.23	−5.70	−4.82

scheme (besides not applying QEC at all) while all other schemes are about equal. When σ_x errors are dominant not applying QEC at all is by far the best scheme. For the other schemes, however, the situation becomes more complex. For small values of p_x there is a general trend of increasing fidelity when applying less QEC. As p_x and the asymmetry increase applying QEC more often becomes optimal. However at even higher degrees of asymmetry applying QEC less often is better. The reason that no QEC is optimal for this error model is because the QEC code plus syndrome measurement used here are more sensitive σ_x errors than other types of errors. The corrective abilities of QEC are not enough to overcome these sensitivities.

In conclusion, we have numerically explored the question of how often quantum error correction needs to be applied during a sequence of logical single-qubit gates from the gate set Clifford plus T, as would be necessary for the implementation of arbitrary single-qubit rotations in a fault tolerance setting. Our simulations involved 50 encoded gates applied to a logical qubit of the [[7,1,3]] QEC code in a non-equiprobable error environment. We demonstrated that for very small depolarization it is best to apply QEC after every gate but for stronger errors QEC should be applied before every T-gate. When the errors are asymmetric the optimal choice of how many times to apply QEC will depend on which error is dominant and the size of the asymmetry. Perhaps surprisingly, when bit-flip errors are dominant it is best not to apply QEC at all. In all cases, however, the difference between applying the QEC scheme with the highest fidelity and applying QEC just once after all 50 gates is minimal. Thus, applying QEC once can lead to a savings of more than an order of magnitude with a negligible cost in fidelity.

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