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ATMOSPHERIC DENSITY MODEL ERRORS AND VARIATIONS IN THE BALLISTIC COEFFICIENT

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Atmospheric density model errors are absorbed as variations in the ballistic coefficient or B term by the differential correction process. Unmodeled forces (e.g., geopotential terms) can also lead to variations in B. For satellites with small energy dissipation rates, observability problems also contribute to variations in B. Monte Carlo simulation is used to determine the accuracy of the variation in the ballistic coefficient in absorbing atmospheric density model errors. The standard deviation of the relative change in B from the least squares differential correction covariance matrix is shown to provide an estimate of the accuracy of the variation of B in absorbing atmospheric density model errors. The accuracy of the variation of B depends on the accuracy of the sensor measurements, the differential correction fit span, and the number of independent observations in the fit span. The length of the fit span is most critical for satellites with B term observability problems.

INTRODUCTION

Snow and Liu[1] discovered that the empirical ballistic coefficients of several satellites exhibited noticeable variations in a synchronized fashion over the same periods of time. They attributed these variations to atmospheric neutral density model errors, and possibly to other model errors, including unmodeled geopotential terms. Marcos, *et. al.,*[2] used the variation in the ballistic coefficient of a satellite to back out atmospheric density model errors. The density model errors were typically about 15%. Time-dependent global corrections to the atmospheric density model were then applied to the orbit determination of other satellites. The corrected atmospheric density model significantly improved the orbit determination of these satellites.

Monte Carlo simulation is a technique that can be used to investigate the statistical variations in ballistic coefficient due to atmospheric density model errors. The advantages of simulation techniques are that one has control over the error sources and their statistical properties, and a "truth" reference orbit is available for comparison with the simulated orbits. The disadvantage of simulation is that the real world may be over simplified so that the results of the simulation are not applicable. Monte Carlo simulation is used in this paper to determine the accuracy of the variation in the ballistic coefficient

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or B term in absorbing atmospheric density model errors. Variations in B due to unmodeled forces are eliminated by accepting a truncated gravitational potential as the truth model. Sensor measurement errors and atmospheric density model errors are the only source of error in the simulation. With these assumptions, the standard deviation of the relative change in B from the least squares differential correction covariance matrix is shown to provide an estimate of the accuracy of variation of B in absorbing atmospheric density model errors. The accuracy of the variation of B depends on the accuracy of the sensor measurements, the differential correction fit span, and the number of independent observations in the fit span. The length of the fit span is most critical for satellites with B term observability problems.

MODEL DESCRIPTION

The equations of motion for a satellite in the earth's gravitation field and atmosphere are given by

(1) $d^2\mathbf{r}/dt^2 = \mathbf{a}_G + \mathbf{a}_D = \nabla U - \frac{1}{2} B\rho v_{rel}\, \mathbf{v}_{rel}$,

where $\mathbf{r}$ is the earth centered inertial (ECI) position vector of the satellite, $\mathbf{a}_G$ is the gravitational acceleration, $\mathbf{a}_D$ is the drag acceleration, U is the gravitational potential, B (m^2/kg) is the ballistic coefficient or B term, ρ (kg/m^3) is the atmospheric density, and $\mathbf{v}_{rel}$ is the velocity of the satellite relative to the atmosphere. For an atmosphere corotating with the earth

(2) $\mathbf{v}_{rel} = \mathbf{v} - \boldsymbol{\omega}\times \mathbf{r}$,

where $\mathbf{v}$ is the ECI velocity vector of the satellite and $\boldsymbol{\omega}$ is the earth's rotation vector. We take the truncated gravitation potential given by the two-body potential plus the J2 term as the true gravitational model. Thus

(3) $U = (\mu/r)[1 + J2(R/r)^2(1 - 3 \sin^2\phi)/2]$,

where μ is the earth's gravitational constant, $\phi = z/r$ is the geocentric latitude, and R is the earth's equatorial radius. Similar to simplifications made by Brouwer and Hori[3], we assume a stationary atmosphere, so that $\mathbf{v}_{rel} = \mathbf{v}$.

We consider satellites in near-circular orbits and make the simplifying assumption that the atmospheric model density, ρ, is constant over the small altitude variations of the satellite. We also assume that that there is a known true density, ρ_{true}, which is constant over the small altitude variations of the satellite. For each Monte Carlo trial for a given satellite, we take the model density, ρ, to be a sample from the normal distribution with mean equal to ρ_{true} and standard deviation equal to 0.15 ρ_{true} (15% error). Thus, ρ is also constant over the differential correction fit span. We assume that the true ballistic coefficient, B_{true}, is known. We then define an ideal ballistic coefficient,

(4) $B_{ideal} = B_{true} (\rho_{true}/\rho)$.

Since $B_{ideal}\ \rho = B_{true}\ \rho_{true}$, the orbit obtained by integrating Eq. (1) with $B = B_{ideal}$ and the simulated ρ is identical to the truth orbit obtained by integrating Eq (1) with B_{true} and ρ_{true} (assuming the initial state vectors are the same). Thus, B_{ideal} is the ballistic coefficient that perfectly absorbs the atmospheric density error in ρ. If the sensor measurements are perfect, then the differential correction process should obtain B_{ideal} as the solved for ballistic coefficient. Assuming that the sensors have some measurement errors, the solved for ballistic coefficient, B, should be close to B_{ideal}. The ratio $(B - B_{ideal})/B_{ideal}$ is the measure of the relative accuracy of B in absorbing the errors in the model density, ρ. We will show that the standard deviation, $\sigma_{\Delta B/B}$, obtained from the covariance matrix in the differential correction process, is an estimate of the relative error $(B - B_{ideal})/B_{ideal}$.

DIFFERENTIAL CORRECTIONS

Differential corrections to the 7 state system consisting of the state vector ($\mathbf{r}$, $\mathbf{v}$) and the ballistic coefficient B are obtained by solving the equation

(5) $Ax = b$

by weighted least squares. The differential correction vector, x, is the column vector with components ($\Delta\mathbf{r}$, $\Delta\mathbf{v}$, $\Delta B/B$). We solve for the relative correction of B rather than the absolute correction. The right-hand side of Eq. (5) is the column vector, b, of the differences between m observed sensor measurements and the calculated measurements from the predicted state obtained by integrating Eq. (1). The left hand side of Eq. (5) is the $m \times 7$ matrix A, obtained as the product of the partial derivatives of the measurements with respect to the state variables times the state transition matrix. The state transition matrix transforms differential corrections at the time of the measurements to the epoch time of the updated state, which we take to be the time of the last observation in the fit span. The state transition matrix satisfies a first order differential equation, which is numerically integrated simultaneously with Eq. (1), rewritten as a first order differential equation. Thus, we are using Cowell's method[4] of special perturbations. A fourth-order Runge-Kutta integrator is used for the integration.

We multiply both sides of Eq. (5) by the $m \times m$ diagonal matrix D, whose diagonal entries are the reciprocals of the standard deviations of the sensor measurement errors. The optimal solution of the weighted least squares problem $(DA)x = Db$ satisfies the normal equations

(6) $(A^TWA)x = A^TWb$,

where $W = D^T D$ and the superscript T indicates the transpose of the matrix. Eq. (6) is solved for the differential correction vector, x, and the process is iterated until the updated state converges. The covariance matrix, C, is given by

(7) $C = (A^T W A)^{-1}$.

Four sensors are positioned on the equator at 0, 90, 180, and 270 degrees latitude. Range measurements are simulated by adding normally distributed errors to the truth orbit. We simulate range measurements at uniform time intervals as long as the satellite is in the sensors' field of view. We assume that the standard deviations of the range errors of all the sensors are the same and have the value σ. Thus, $W = \sigma^{-2} I$, where I is the identity matrix.

SIMULATION RESULTS

Consider a satellite with $B_{true} = 0.01\ m^2/kg$ in a near-circular orbit at an altitude of 400 km and an inclination of 60 deg. This is a typical value for the ballistic coefficient for payloads and rocket bodies. We take the true density, ρ_{true}, at an altitude of 400 km to be $3.725 \times 10^{-12}\ kg/m^3$, and assume it is constant over small variations in altitude so that the satellite experiences a constant atmospheric density. This density value was taken from an exponential atmospheric model[5], but we are not using the exponential variation of density with altitude. For each Monte Carlo trial, we take the initial state vector to be the truth state vector and the ballistic coefficient $B = B_{true}$. Each Monte Carlo trial uses a simulated value of ρ as a sample from the normal distribution with mean ρ_{true} and standard deviation 0.15 ρ_{true}, and a set of sensor range measurements with zero mean and standard deviation σ. The differential correction process will correct the state vector and ballistic coefficient B to fit the simulated density ρ and sensor measurements.

The energy dissipation rate (EDR) of a satellite is defined by the path integral

(8) $$\text{EDR} = -(1/T) \int_{s1}^{s2} \mathbf{a}_D \bullet d\mathbf{s},$$

where T is the time between the points **s**1 and **s**2, which we take to be the differential correction fit span, and d**s** is the differential arc length along the orbit. For the above satellite, EDR = 0.008408 (W/kg), which corresponds to a moderate amount of drag.

Using the truth orbit of the satellite, we compute the sample standard deviations ($s_{\Delta u}$, $s_{\Delta v}$, $s_{\Delta w}$) of the (Δu, Δv, Δw) position errors at the epoch time from n Monte Carlo trials, where (u, v, w) is the radial, in-track, and cross-track coordinate system relative to the satellite. We also compute the sample standard deviation $s_{\Delta B/B}$ of $(B - B_{ideal})/B_{ideal}$.

For each Monte Carlo trial, the 3×3 spatial part of the covariance matrix C is rotated into the (u, v, w) coordinate system to obtain the standard deviations ($\sigma_{\Delta u}$, $\sigma_{\Delta v}$, $\sigma_{\Delta w}$). The square root of the (7,7) entry of C is $\sigma_{\Delta B/B}$. We then take the sample mean and sample standard deviation of these covariance derived sigmas. A 100(1 - α)% confidence interval for the standard deviation is given by

(9) $((1\text{-}n)^{1/2}s/(\chi^2_{1-\alpha/2,\, n-1})^{1/2},\ (1\text{-}n)^{1/2}s/(\chi^2_{\alpha/2,\, n-1})^{1/2})$,

where s is the sample standard deviation, n is the number of samples, and $\chi^2_{p,\, n}$ is the (100p)th percentile of the χ^2 distribution with n degrees of freedom. The level of significance is α, which we will take as 0.05.

Table 1 shows the statistics for the Monte Carlo simulation for the above satellite with n = 1000, σ = 10.0 m, a 1-day fit span, and a 60 sec interval between sensor observations. The sample standard deviations of the covariance derived sigmas are extremely small, implying that these sigmas are essentially constant between Monte Carlo trials. This is due to the fact that the sensors' range sigma, the fit span, and the interval between sensor observations are kept constant for the Monte Carlo trials. For each variable, the means of the covariance derived sigmas all fall within the 95% confidence interval for the standard deviations obtained from Eq. (9). Thus, each of these sigmas is a good estimate of the standard deviation of the errors.

Table 1

ERROR STATISTICS FOR SATELLITE AT 400 KM ALTITUDE

Variable	Sample mean	Sample standard deviation	95% confidence interval for the standard deviation	Sample mean of covariance sigma	Sample standard deviation of covariance sigma
Δu (m)	-.09	2.68	(2.56, 2.80)	2.64	.000001
Δv (m)	.09	3.65	(3.41, 3.73)	3.53	.000001
Δw (m)	-.01	2.38	(2.28, 2.49)	2.46	.000003
$(B - B_{ideal})/B_{ideal}$	-.00006	.00116	(.00111,.00122)	.00120	.000001

The relative variation of B from B_{ideal} is very small, 0.0012 or 0.12%. However, the relative variation (sample standard deviation of $(B - B_{true})/B_{true}$) of B from B_{true} is large, 0.165 or 16.5%, which is larger than the 15% error in ρ. The sample mean of B is 0.01028, which is a 2.8% error from B_{true}. Because we assumed that the errors in ρ are normally distributed and B is approximately equal to B_{ideal}, Eq. (4) implies that B is not normally distributed. The median is a more robust estimate of the central tendency of a distribution than the mean. We note that the median of B is 0.01010, which is a 1.0% error from B_{true}. From Eq. (4), $1/B_{ideal} = \rho/(B_{true}\ \rho_{true})$. Since B is approximately equal to B_{ideal}, we expect that 1/B is approximately normally distributed with mean $1/B_{true}$ = 100.0 and standard deviation equal to 15.0. The sample mean of 1/B is 99.60 and the sample standard deviation of 1/B is 14.83. We note that 1/(mean(1/B)) = 0.01004, which is a 0.4% error from B_{true}. The Kolmogorov-Smirnov test[6] is applied to 1/B and the normal distribution with mean 100.0 and standard deviation 15.0. Figure 1 shows the cumulative distribution of 1/B and the cumulative normal distribution. The maximum vertical distance, d, between the two graphs is 0.032, which corresponds to a probability of 0.248 from the Kolmogorov-Smirnov test with 1000 samples. At the level of significance of 0.05, we cannot reject the null hypothesis that 1/B is normally distributed with mean 100.0 and standard deviation 15.0.

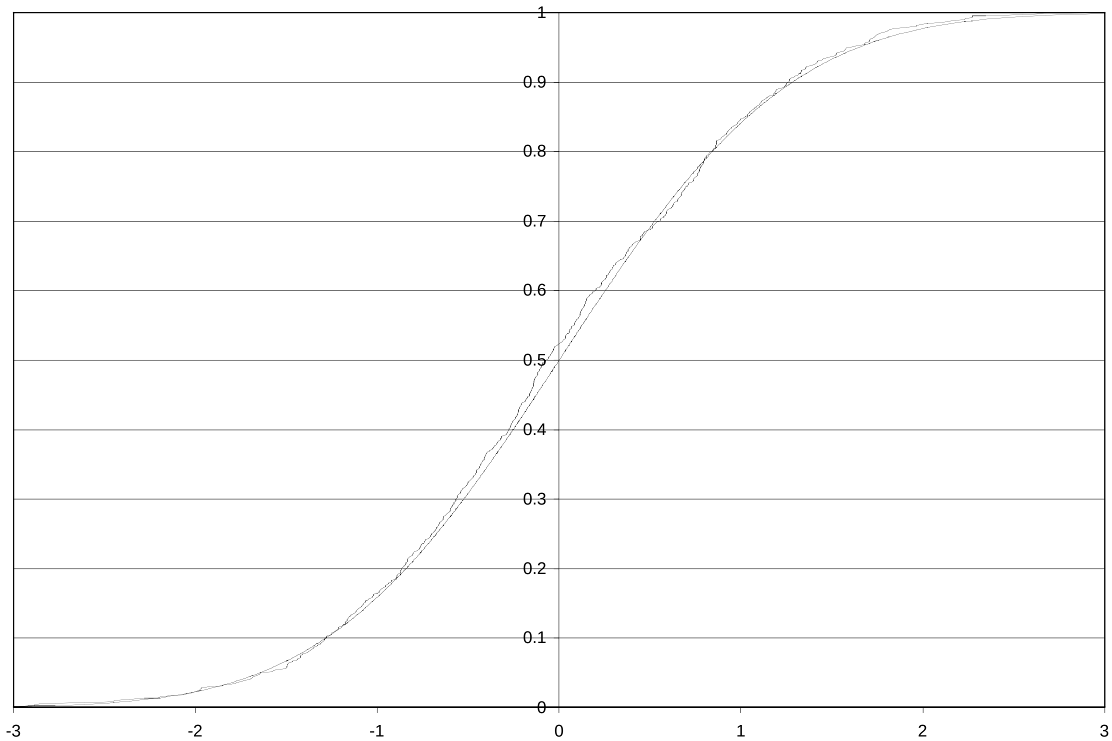

Figure 1 Cumulative Distribution of 1/B and the Cumulative Normal Distribution

From an examination of empirical ballistic coefficients of many satellites during the year 2000 obtained from special perturbation differential corrections of real world sensor observations, neither B nor 1/B appears to be normally distributed. It is therefore advisable to use the median of B as the best estimate of the true value of the ballistic coefficient. Note that median(B) = 1/(median(1/B)). In general, if f(X) is a monotonic transformation of a random variable X, then median(f(X)) = f(median(X)), i.e., the median is preserved under a monotonic transformation. The mean of a random variable is not preserved under a monotonic transformation.

Table 2 shows other Monte Carlo simulations for the same satellite with different values for σ, fit spans, and time intervals between sensor observations. Each Monte Carlo simulation in Table 2 contains 1000 trials. All the covariance derived sigmas are within the 95% confidence interval for the standard deviations and are thus good estimates of the standard deviations. The second row of Table 2 is the Monte Carlo simulation from Table 1. Since $W = \sigma^{-2} I$, Eq. (7) implies that C is proportional to σ^2 and the covariance derived sigmas are proportional to σ. The first three rows of Table 2 show this linear dependence on σ. Rows 1, 4, and 5 of Table 2 show that the covariance derived sigmas are approximately proportional to $1/\sqrt{n}$, where n is the number of sensor measurements. This $1/\sqrt{n}$ relationship between the sigmas and the number of sensor measurements is a result of the assumption that the sensor measurements are statistically independent. Real world sensor measurements may not be independent when the time interval between observations is short, in which case we would need to simulate the observations in a track as an auto-correlated time series instead of independent measurements. Rows 1, 6, and 7 of Table 2 show that the covariance derived sigmas for the position errors are fairly constant for different fit spans, provided the number of sensor measurements remains constant. (There is a slight increase in $\sigma_{\Delta v}$ and a slight decrease in $\sigma_{\Delta w}$ as the fit span increases.) However, $\sigma_{\Delta B/B}$ is approximately proportional to $1/n^2$, where n is the number of days in the fit span.

Table 2

MONTE CARLO SIMULATIONS FOR SATELLITE AT 400 KM ALTITUDE

σ (m)	Fit span (days)	Time interval between sensor observations (sec)	Number of sensor measurements	$\sigma_{\Delta u}$ (m)	$\sigma_{\Delta v}$ (m)	$\sigma_{\Delta w}$ (m)	$\sigma_{\Delta B/B}$
100	1	60	131	26.36	35.28	24.60	.0120
10	1	60	131	2.64	3.53	2.46	.00120
1	1	60	131	0.26	0.35	0.25	.000120
100	1	30	265	18.25	25.12	18.53	.00844
100	1	10	794	10.40	14.52	11.39	.00486
100	2	120	126	26.93	40.98	23.37	.00344
100	3	180	128	26.65	43.22	19.01	.00162

Now consider the same satellite with B_{true} = 0.01 m^2/kg in a near circular orbit at an altitude of 800 km and an inclination of 60 deg. We take $\rho_{true} = 1.170 \times 10^{-14}$ kg/m^3, and assume it is constant over the small variations in the altitude of the satellite. This density was taken from the exponential atmospheric density model at an altitude of 800 km from Table 7-4 in Ref. 5. This same satellite at an altitude of 800 km has an EDR equal to 0.00002423 W/kg, which corresponds to a very small amount of drag.

For each Monte Carlo trial we again take the model density, ρ, to be a sample from the normal distribution with mean equal to ρ_{true} and standard deviation equal to 15% of ρ_{true}. Table 3 shows the statistics for the Monte Carlo simulation for this satellite with n = 1000, σ = 100.0 m, a 3-day fit span, and a 60 sec interval between sensor observations. All the covariance derived sigmas for the position errors are nearly constant between Monte Carlo trials, and their means fall within the 95% confidence interval for the standard deviations. However, the covariance derived $\sigma_{\Delta B/B}$ is not constant between Monte Carlo trials (its sample standard deviation is 0.095), and the sample mean of $\sigma_{\Delta B/B}$ is not in the 95% confidence interval for the standard deviation. There is an observability problem for this ballistic coefficient at this altitude with these sensor measurement errors and fit span. The differential corrections for B are fitting sensor measurement noise and are unable to correct B to B_{ideal}.

Table 3

ERROR STATISTICS FOR SATELLITE AT 800 KM ALTITUDE

Variable	Sample mean	Sample standard deviation	95% confidence interval for the standard deviation	Sample mean of covariance sigma	Sample standard deviation of covariance sigma
Δu (m)	-.24	8.34	(7.99, 8.73)	8.26	.000009
Δv (m)	-.18	19.59	(18.77, 20.49)	19.58	.000019
Δw (m)	.21	10.67	(10.22, 11.16)	10.42	.000049
$(B - B_{ideal})/B_{ideal}$	-.0133	.245	(.235,.256)	.270	.095

This observability problem can be overcome by decreasing the sensor measurement errors or increasing the fit span. Increasing the number of observations only marginally improves the observability of B. Table 4 shows other Monte Carlo simulations for this satellite with different values for σ, fit spans, and time intervals between sensor observations. Each Monte Carlo simulation in Table 4 contains 1000 trials. The first row of Table 4 is the Monte Carlo simulation from Table 3. All the covariance derived sigmas are within the 95% confidence interval for the standard deviations except $\sigma_{\Delta B/B}$ in the first row. The first three rows of Table 4 show the linear dependence on σ. The sample standard deviation 0.245 is a better estimate of the standard deviation than $\sigma_{\Delta B/B}$ for the first row and better fits the linear trend with σ. Rows 1, 4, and 5 of Table 4 show that the covariance derived sigmas are approximately proportional to $1/\sqrt{n}$, where n is the number of sensor measurements. Rows 1, 6, 7, and 8 of Table 4 show that the covariance derived sigmas for the position errors are fairly constant for different fit spans, provided the number of sensor measurements remains constant. (There is a slight decrease in $\sigma_{\Delta w}$ as the fit span increases until the fit span equals 12 days, in which case there is a slight increase in $\sigma_{\Delta w}$.) Again, $\sigma_{\Delta B/B}$ is approximately proportional to $1/n^2$, where n is the number of days in the fit span. The sample standard deviation 0.245 is a better estimate of the standard deviation than $\sigma_{\Delta B/B}$ for the first row and better fits the $1/n^2$ trend with the length of the fit span.

Table 4

MONTE CARLO SIMULATIONS FOR SATELLITE AT 800 KM ALTITUDE

σ (m)	Fit span (days)	Time interval between sensor observations (sec)	Number of sensor measurements	$\sigma_{\Delta u}$ (m)	$\sigma_{\Delta v}$ (m)	$\sigma_{\Delta w}$ (m)	$\sigma_{\Delta B/B}$
100	3	60	727	8.26	19.58	10.42	.270
10	3	60	727	.83	1.96	1.04	.0247
1	3	60	727	0.08	0.20	0.10	.00246
100	3	30	1453	5.83	13.84	7.37	.181
100	3	10	4367	3.35	8.02	4.32	.102
100	6	120	737	8.34	20.42	7.52	.0630
100	9	180	741	8.49	20.11	5.86	.0280
100	12	240	724	8.76	20.27	6.25	.0163

CONCLUSIONS

It has been shown that the covariance derived $\sigma_{\Delta B/B}$ is a good estimate for the standard deviation of $(B - B_{ideal})/B_{ideal}$, provided there are no observability problems for the ballistic coefficient, B. If there are no observability problems for B, $\sigma_{\Delta B/B}$ is small (on the order of 1% or less). The standard deviation of $(B - B_{true})/B_{true}$ can be quite large and depends on the errors in the atmospheric model density, ρ. It has also been shown that the covariance derived sigmas for the position errors and $\sigma_{\Delta B/B}$ have specific functional dependencies on the sensor measurement errors, differential correction fit span, and number of independent observations in the fit span.

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