# A Steady State Decoupled Kalman Filter Technique for Multiuser Detection

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## Abstract

*In this paper, we describe a Kalman filter based technique for multiuser detection of asynchronous CDMA systems. Similar to previously proposed techniques, we use a decoupled form of the Kalman filter which processes each user independently. We then make the further simplifying assumption that the channel statistics are constant over a single symbol allowing us to reduce the kalman gain to a constant dependent on the current signal to interference plus noise ratio. We study the performance of the constant gain filter and show that by appropriate choice of the gain you can optimize for either minimum bit error rate or maximum interference reduction.*

## 1. Introduction

Code division multiple access (CDMA) communication systems are commonly limited in performance by multiple access interference (MAI). This interference is caused by less than perfect orthogonality between multiple simultaneous users all transmitting asynchronously. This problem will only get worse over time as the number of wireless users continues to increase. Many multiuser detection methods have been developed to deal with MAI, but receiver complexity often limits their practical utility.

In this paper, we will be specifically investigating a Kalman filter based technique. Standard Kalman filter approaches have been shown to significantly outperform the conventional receiver, but at high computational complexity [4]. A suboptimal technique was proposed in [1],[2] which made use of a decoupled form of the Kalman filter to significantly reduce the computational cost with only a minor degradation in performance. We have extended this technique by observing that since the Kalman filter is reset after every symbol, we only care about the channel statistics during the very short interval of a single symbol. By assuming that the statistics are roughly stationary during that interval, we can use the steady state solution to the Kalman filter and replace the Kalman gain with a constant. Simulations show that this assumption has little effect on performance.

In Section 2 we define the problem and our signal model. In Section 3 we derive the steady state Kalman filter solution to the problem and consider other potential choices for the fixed gain constant. Section 4 analyzes the decoupled Kalman filter and derives a prediction of its performance. This prediction also holds for the fixed gain formulation. Finally, simulation results are presented in Section 5.

## 2. Problem Formulation

We consider an asynchronous CDMA system using binary phase shift keying modulation and operating over an additive white Gaussian noise channel. $K$ active users are received simultaneously. The received signal is then

$$\mathbf{y}(t) = \sum_{k=1}^{K} \mathbf{h}_k(t-\tau_k)\mathbf{a}_k(t)\mathbf{b}_k(t) + \mathbf{n}(t) \qquad (1)$$

Where $\mathbf{y}(t)$ is the received signal, $\mathbf{h}_k(t-\tau_k)$ is the spreading function for user $k$, $\tau_k$ is the delay, $\mathbf{a}_k(t)$ is the complex channel amplitude, $\mathbf{b}_k(t)$ is the transmitted bit ($\pm 1$), and $\mathbf{n}(t)$ is the background additive noise.

The user time delays $\tau_k$ and channel amplitudes and phases $\mathbf{a}_k$ are slowly varying and are estimated external to the algorithm using early/late gates or phase-locked-loop techniques applied to pilot channels. In typical CDMA signals (i.e. IS95) the user traffic channels are transmitted at the same delay and phase as the pilot channels, but at different amplitudes (usually 6dB or so weaker than the pilot). The amplitudes of each of the user channels has to be estimated to correctly cancel the interference.

## 3. Steady State Decoupled Kalman Filter

Similar to [1] and [2] we exploit the pseudorandom independence of the spreading codes and develop separate Kalman filters for each user. The received signal model for user $k$ is then

$$\mathbf{y}_k(t) = \mathbf{h}_k(t)\mathbf{a}_k(t)\mathbf{b}_k(t) + \mathbf{v}(t) \tag{2}$$

Where $\mathbf{v}(t)$ now includes the interference from all other users which through use of the central limit theorem we approximate as a zero mean gaussian with unknown variance $\frac{1}{2}\sigma_v^2(t)$.

Letting $\mathbf{x}_k(t) = \mathbf{a}_k(t)\mathbf{b}_k(t)$, we can then write the state equations as

$$\begin{aligned} \mathbf{x}_k(t+1) &= \Phi_{kk}(t)\mathbf{x}_k(t) + \mathbf{w}_k(t) && (3) \\ \mathbf{y}_k(t) &= \mathbf{h}_k(t)\mathbf{x}_k(t) + \mathbf{v}(t) && (4) \end{aligned}$$

Where the state transition matrix $\Phi_{kk}(t)$ is equal to 1 except when there is a symbol change between times $t-1$ and $t$ at which point it is set to zero. Similarly, the state noise $\mathbf{w}_k(t)$ is zero except where there is a symbol change. The variance of $\mathbf{w}_k(t)$ during a symbol change is $Q_k(t)$ and is equal to $E(a_k(t)^* a_k(t))$.

We can now write the Riccati equations and the state equation updates. The Riccati equations are

$$\begin{aligned} P_{kk}(t)^- &= \Phi_{kk}(t)P_{kk}(t-1)^+\Phi_{kk}(t)^* + Q_k(t) && (5) \\ K_k(t) &= P_{kk}(t)^- h_k(t)^*[h_k(t)P_{kk}(t)^- h_k(t)^* \\ &\quad + \frac{1}{2}\sigma_v^2(t)]^{-1} && (6) \\ P_{kk}(t)^+ &= [1 - K_k(t)h_k(t)]P_{kk}(t)^- && (7) \end{aligned}$$

And the state equation updates are

$$\begin{aligned} x_k(t)^- &= \Phi_{kk}(t)x_k(t-1)^+ && (8) \\ x_k(t)^+ &= x_k(t)^- + K_k(t)\left[y_k(t) - h_k(t)x_k(t)^-\right] && (9) \end{aligned}$$

The nature of the state model causes the Kalman filter to reset after every symbol change. This means that the quality of the bit estimate is only dependent on the measured data within a single symbol. This severely limits the amount of gain we can hope to get from additional processing. However, we can use this limitation to our advantage. Since our estimate is only dependent on the data statistics within a single symbol (which is typically of very short duration), we can assume that the channel statistics are essentially constant. With everything constant, we can make use of the steady state solution to the Kalman filter and reduce the Kalman gain to a constant.

Combining (5, 6, and 7) we know that the steady state covariance $P_{kk}(\infty)$ satisfies the following

$$\begin{aligned} P_{kk}(\infty) &= \Phi_{kk}\{P_{kk}(\infty) - P_{kk}(\infty)h_k^* \\ &\quad \left[h_k P_{kk}(\infty)h_k^* + \frac{1}{2}\sigma_v^2\right]^{-1} \\ &\quad h_k P_{kk}(\infty)\}\Phi_{kk}^* + Q_k && (10) \end{aligned}$$

Solving for $P_{kk}(\infty)$ reveals a problem related to our choice for the process noise covariance $Q_k$. Earlier we had said that $Q_k$ was zero during a symbol and was $E(a_k(t)^* a_k(t))$ when a symbol change occurred. Which value should we use then for solving for the steady state Kalman gain? Using zero results in a Kalman gain of zero, so that definitely doesn't work. The alternative results in a non-zero gain, but implies a very large process noise error at every state update. We instead chose to average the process noise over an entire symbol. This accurately captures all of the expected process noise while still preserving our steady state model. The process noise covariance is then

$$Q_k = \frac{1}{N}E(a_k(t)^* a_k(t)) \tag{11}$$

where $N$ is the coding gain of the PN sequence. Solving (10) for $P_{kk}(\infty)$ now yields a quadratic with the roots

$$\frac{1}{2}\left(Q_k \pm \sqrt{Q_k^2 + \frac{2Q_k\sigma_v^2}{h_k h_k^*}}\right) \tag{12}$$

The steady state Kalman gain is then

$$K_{kk}(\infty) = P_{kk}(\infty)h_k^*\left(h_k P_{kk}(\infty)h_k^* + \frac{1}{2}\sigma_v^2\right)^{-1} \tag{13}$$

Where $P_{kk}(\infty)$ is the positive root from (12). Unsurprisingly, the constant Kalman gain is strongly related to the current signal to interference plus noise ratio (SINR). Which we define as

$$SINR = \frac{2|a_k|^2}{\sigma_v^2} \tag{14}$$

Comparing the constant gain solution to the standard decoupled Kalman filter, we see that the constant gain solution reduces the number of complex multiplies required at each innovation update from 8 to 2. This solution is equivalent to a Weiner filter and we refer to it as the Weiner Filter Demodulator (WFD).

Smearing out the measurement noise across an entire symbol is not entirely satisfactory however. We should also consider other choices for the constant gain and try to understand how they will effect performance. In Sections 4 and 5 we analyze the performance of the Kalman and fixed gain algorithms and simulate their performance using a wide variety of gain values.

In any case, to implement this technique we require some method for estimating the multiple access interference (MAI), $\frac{1}{2}\sigma_v^2$, as well as the individual channel magnitudes

$|a_k|^2$. The simplest way to do this is directly from the received data. We calculate the sample variance of the data as an initial estimate of the MAI. We then run multiple passes of the algorithm and update the estimate using the sample variance of the residual error after each pass. This directly captures the interference cancellation provided by the algorithm. Similarly, we use the pilot channel magnitudes as an initial estimate of the individual traffic channel magnitudes. After each pass of the algorithm, we calculate the average magnitude of the bit estimates for each user and use them as the channel magnitude estimates for the next pass.

## 4. Performance Prediction

We would like to have a closed form prediction of decoupled Kalman filter performance (with or without a fixed gain). Unfortunately, this is very difficult to do. However, an iterative solution is fairly straightforward. If we consider only a single symbol (i.e. $0 \le t < T$), the standard predictor-corrector structure of the Kalman filter is the following:

$$\begin{aligned} \hat{x}^-(0) &= 0 \\ \hat{x}^-(t) &= \hat{x}^+(t-1) \\ \hat{x}^+(t) &= \hat{x}^-(t) + k(t)h(t)^*(y(t) - h(t)\hat{x}^-(t)) \end{aligned} \quad (15)$$

The state estimate $\hat{x}^+(T-1)$ provides the final bit estimate. The goal of our Kalman filter is to choose, for each time $t$, a gain $k(t)$ that minimizes the mean squared error

$$E(|\hat{x}^+(t) - x(t)|^2) \quad (16)$$

First introduce the notation

$$\begin{aligned} P^+(t) &= E(|\hat{x}^+(t) - x(t)|^2) \\ P^-(t) &= E(|\hat{x}^-(t) - x(t)|^2) \end{aligned} \quad (17)$$

Clearly,

$$\begin{aligned} P^-(0) &= 1 \\ P^-(t) &= P^+(t-1) \\ P^+(t) &= E(|\hat{x}^+(t) - x(t)|^2) \\ &= E(|\hat{x}^-(t) - x(t) + k(t)(-h(t)^*h(t)(\hat{x}(t)^-) \\ &\quad -x(t)) + u(t))|^2) \\ &= P^-(t) - k(t)h(t)^*h(t)P^-(t) \\ &\quad -k(t)^*h(t)h(t)^*P^-(t) \\ &\quad +k(t)^*k(t)(h(t)^*h(t))^2P^-(t) \\ &\quad +k(t)^*k(t)\sigma_u^2 \end{aligned} \quad (18)$$

We can now address the performance of the Kalman filter by noting that the filter is linear, so the conditional distribution of $\hat{x(t)}$ given $b = x(t) = \pm 1$ is Gaussian within the constraint of the Central Limit Theorem approximation for uncancelled co-channel interference. We can write conditional distributions for the mean and variance of $\hat{x}(t)$ in exactly the same way as in (15).

$$\begin{aligned} E(\hat{x}^-(0)|b) &= 0 \\ E(\hat{x}^+(t)|b) &= E(\hat{x}^-(t)|b) + k(t)h(t)^*b \\ &\quad -k(t)h(t)^*h(t)E(\hat{x}^-(t)|b) \\ E(\hat{x}^-(t)|b) &= E(\hat{x}^+(t-1)|b) \\ P^-(0|b) &= E((\hat{x}^-(0) - b)^2|b) = 1 \\ P^+(t|b) &= P^-(t|b) - k(t)h(t)^*h(t)P^-(t|b) \\ &\quad -k(t)^*h(t)h(t)^*P^-(t|b) \\ &\quad +k(t)^*k(t)(h(t)^*h(t))^2P^-(t|b) \\ &\quad +k(t)^*k(t)\sigma_u^2 \\ P^-(t|b) &= P^+(t-1|b) \end{aligned} \quad (19)$$

These equations are iteratively solved until the end of the symbol period. Since the conditional and non-conditional error variances have the same initial conditions and have the same iterative equations as the original Riccati equations,

$$\begin{aligned} P^-(t|b) &= P^-(t) \\ P^+(t|b) &= P^+(t) \end{aligned} \quad (20)$$

The conditional estimate $\hat{x}^+(t|b)$ is normally distributed with mean given in equation (19). To calculate the variance, use equation (19) and

$$\begin{aligned} var(\hat{x}^+(t|b)) = &\ E(\hat{x}^{+^2}(t)|b) - (E(\hat{x}^+(t)|b))^2 \\ = &\ P^+(t|b) + 2bE(\hat{x}^+(t)|b) - 1 \\ &\ -(E(\hat{x}^+(t)|b))^2 \end{aligned} \quad (21)$$

If $F(\alpha)$ is the probability that a standard normal random variable is less than $\alpha$, then the bit error rate is

$$BER = F\left(\frac{-bE(\hat{x}^+(t)|b)}{\sqrt{var(\hat{x}^+(t|b))}}\right) \quad (22)$$

The above considers only a single user so the interaction between the filters when multiple users are present is not accounted for. Still, it gives us some insight into the effect of the gain on BER and interference cancellation performance. Figure 1 shows a plot of the predicted performance as well as a direct Monte Carlo simulation of the two demodulators for a single user with a post despread SNR of $E_b/I_o$ = 8 dB. The number of chips per symbol was 64 and $h(t)^*h(t)$ = 100. The top plot is BER, the middle plot is the residual interference, and the bottom plot is the impulse response for various choices of the fixed gain.

The results show the limitations of the fixed gain algorithm when the interference is small. Although the effective interference cancellation nearly equals that of the Kalman for a normalized fixed gain of 0.054, the corresponding BER is more than an order of magnitude larger. Examination of the impulse responses shown at the bottom of figure 1 provides an explanation. The Kalman filter impulse response is constant. Since this then provides a matched filter, the Kalman provides the theoretical minimum BER. The fixed gain technique in comparison, weights information late in the symbol much more heavily that information early in the symbol. This loss of information results in a much larger BER. In a multistage algorithm, however, this may not be a significant issue. The objective of the early stages is to cancel interference, which can be accomplished almost optimally with a constant gain algorithm.

The advantage of the decoupled Kalman technique diminishes as $E_b/I_o$ decreases. The performance is summarized in Table 1. For interference levels typical of heavily loaded systems, the fixed gain technique provides an attractive option for interference cancellation.

| $E_b/I_o$ | Kalman Interference | Fixed Gain Interference | Kalman BER | Fixed Gain BER | Fixed Gain with lowest interference |
|---|---|---|---|---|---|
| 8 dB | 0.20 | 0.25 | 0.00019 | 0.0048 | 0.054 |
| 6 dB | 0.27 | 0.32 | 0.0024 | 0.012 | 0.042 |
| 4 dB | 0.35 | 0.39 | 0.0013 | 0.028 | 0.034 |
| 2 dB | 0.44 | 0.47 | 0.038 | 0.054 | 0.026 |
| 0 dB | 0.54 | 0.56 | 0.079 | 0.090 | 0.018 |

## 5. Simulation Results

We simulated the performance of the fixed gain algorithm for a more realistic scenario in Figure 2. The IS-95 standard was used to generate the signals. Each carrier consisted of a pilot with eight traffic channels. The carriers were a mix of weak and strong signals with 75% of the carriers 6 dB weaker than the strong signals. All traffic channels were also 6 dB weaker than their associated pilots. The background noise level was set so the weakest channels had an SNR of 8 dB. Twelve carriers were used with 8 traffic channels each, corresponding to a normalized capacity of 1.5. All carriers are received asynchronously. BER performance and residual interference are shown versus an array of fixed gain values.

Similar to the theoretical predictions in Figure 1, the points of minimum BER and maximum interference reduction occur at different fixed gains. Minimum BER occurs with a gain of 0.007 and maximum interference reduction is at a gain of 0.022. 0.007 also corresponds to the value for the fixed gain determined by equation (13) for the WFD algorithm for this scenario. In further analysis we have found the WFD algorithm to consistently pick the fixed gain which results in the minimum BER.

Finally, in Figure 3, we directly compare the WFD algorithm and decoupled Kalman demodulator. The input signals were identical to the simulation in Figure 2 except that we varied the number of carriers from 4 to 28, corresponding to loading factors from 0.5 to 3.5. As you can see, the BER performance for the two techniques is virtually identical at all loading factors. We do need to point out, however, that performance at the higher normalized capacities is almost certainly dominated by the channel magnitude estimates which were the same for both demodulation techniques.

## 6. Conclusions

By making use of the steady state solution to the Kalman filter we have reduced the already efficient decoupled Kalman filter technique to a single calculation for each symbol. This simplification causes some performance loss in low interference environments, but very little loss in high interference environments. Additionally the fixed gain can be tuned to optimize for either BER performance or interference cancellation depending on the needs of the application.

## References


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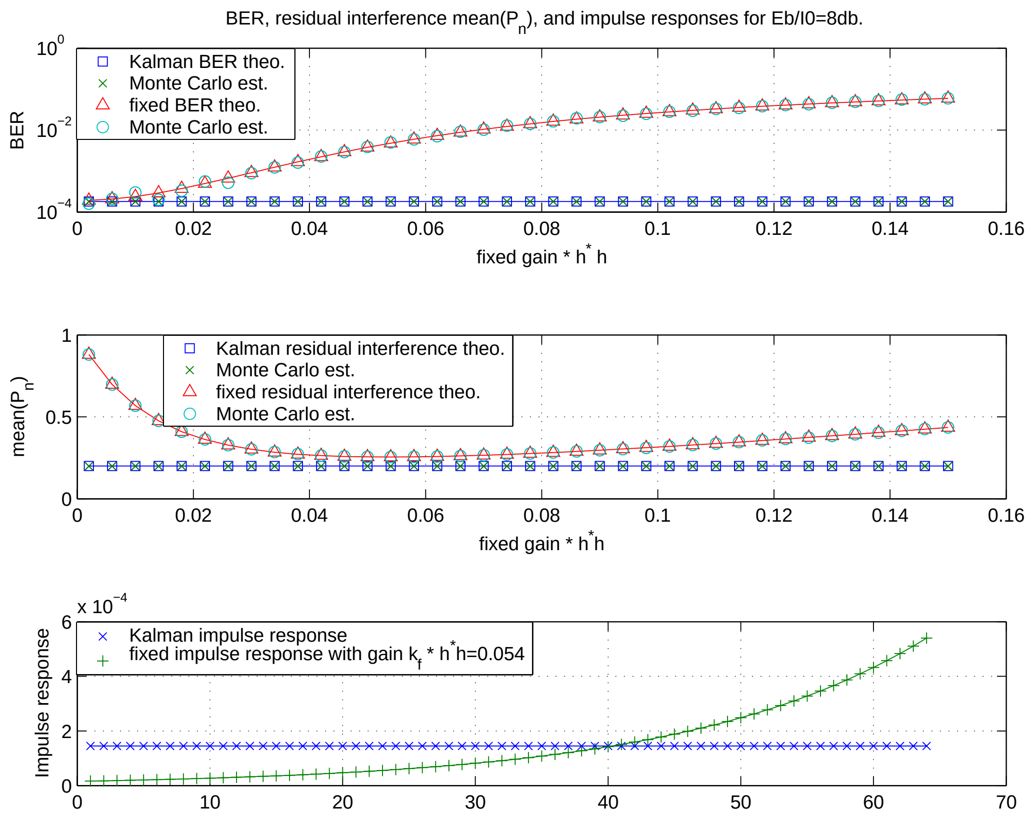


**Figure 1. Comparison of bit error rate (BER) performance, residual interference, and impulse response vs. normalized fixed gain for Kalman and fixed gain demodulators. There were 64 chips per symbol and the SNR was 8 dB.**

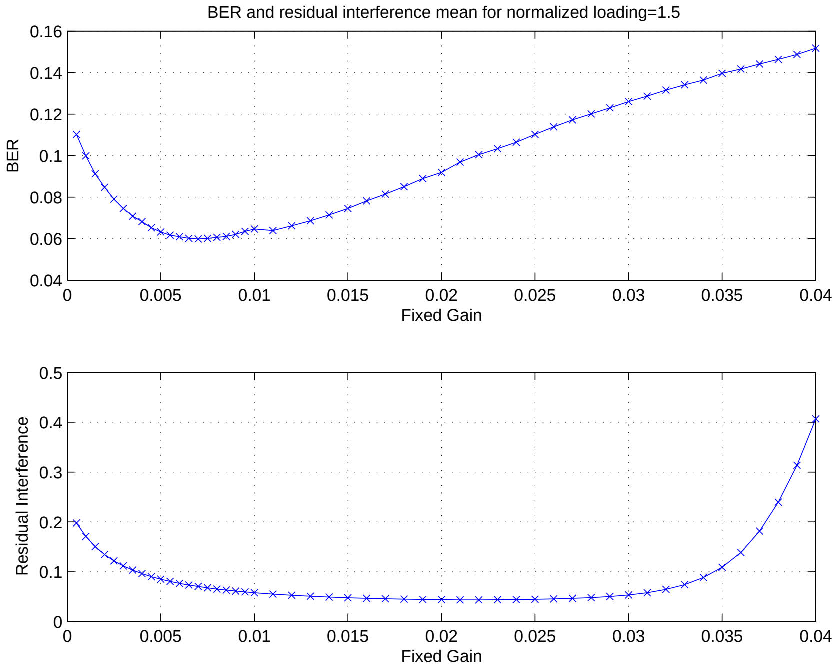


**Figure 2. Comparison of bit error rate (BER) performance and residual interference vs. fixed gain selection at a loading factor of 1.5 and 8 dB SNR.**

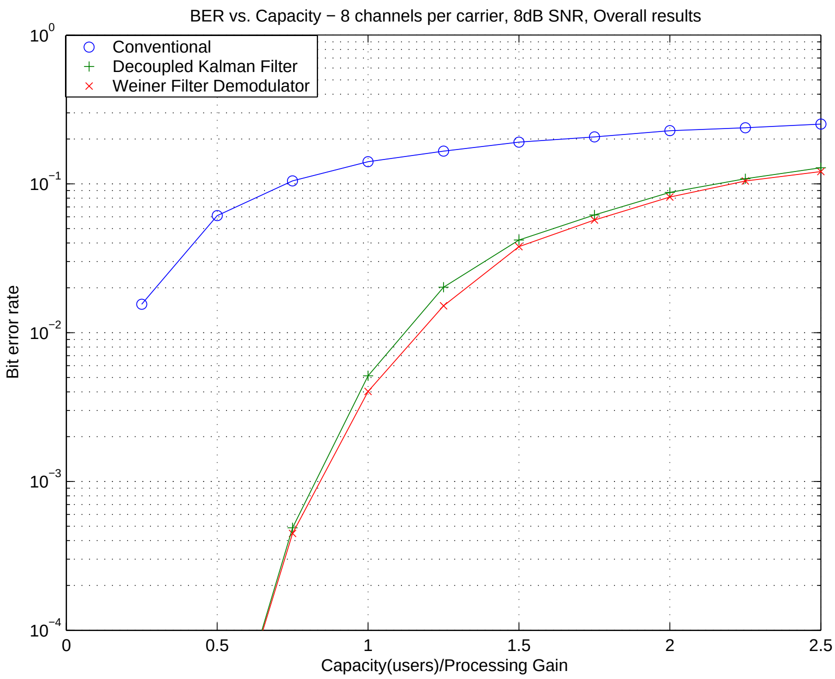


**Figure 3. Comparison of bit error rate (BER) performance vs. normalized capacity at 8 dB SNR.**