How Often Must Quantum Error Correction be Implemented?

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Correcting errors is a vital but expensive component of fault tolerant quantum computation. Standard fault tolerant protocol assumes the implementation of error correction, via syndrome measurements and possible recovery operations, after every quantum gate. In fact, this is not necessary. Here we demonstrate that error correction should be applied more sparingly. We simulate encoded single qubit rotations within the [7,1,3] code and show via fidelity measures that applying error correction after every gate is not desirable. The simulations also shed light on what accuracy can be expected for noisy error correction and thus to what accuracy arbitrary single qubit rotations should be implemented.

PACS numbers: 03.67.Pp, 03.67.-a, 03.67.Lx

Quantum error correction (QEC) [1–3] is a necessary protocol for quantum computation but one that is very expensive in terms of number of qubits required and time to implement. Standard approaches to quantum fault tolerance (QFT), the computational framework that will allow for successful quantum computation despite a finite probability of error in basic computational gates [4–7], nevertheless assume that QEC is applied after every operation. In this paper we demonstrate that applying QEC after every operation is not necessary and in fact should not be done. This assertion is corroborated by simulating multiple single-logical-qubit operations on information encoded in the [7,1,3] QEC code [8].

When implementing gates on encoded information we must ensure that the information does not leave the encoded space such that it may be subjected to errors. For many QEC codes universal quantum computation can be performed without leaving the encoded space if the gate set is restricted to Clifford gates plus the T-gate, a single qubit $\pi/4$ phase rotation. A method for implementing an arbitrary single-qubit rotation (within prescribed accuracy ϵ) with this restricted gate set was initially explored in [9, 10] and has recently become an area of intense investigation [11–17]. For Calderbank-Shor-Steane (CSS) codes, Clifford gates can be implemented bit-wise while the T-gates require a specially prepared ancilla state and a series of controlled-NOT gates. Thus, the primary goal of these investigations has been to construct circuits within ϵ of a desired (arbitrary) rotation while limiting the number of resource-heavy T-gates. As an example, $R_Z(.1)$ can be implemented with accuracy better than 10^{-5} using 78 [16] or 56 [17] T-gates, interspersed by at least as many single qubit Clifford gates. QFT would suggest that QEC be applied after each one of the more than 100 gates needed to implement such a rotation requiring thousands of additional qubits and hundreds of time steps. Thus, adhering to this tenet of QFT is very resource intensive.

Recent work has been devoted to exploring the possibility of relaxing certain tenets of QFT while retaining to the ability to reliably compute [18–20]. In line with this work, we demonstrate that QEC need not be applied after every gate and, in fact, should not be applied after every gate. Applying QEC less often will consume less resources, while still enabling successful quantum computation. Note that by application of QEC we refer to the implementation of syndrome measurements and possible recovery operations that must actively be applied during the computation. The entire computation, however, will of course be performed within a QEC encoding.

The [7,1,3] QEC code will correct an error on one physical qubit out of the seven qubit system in which one qubit of quantum information is encoded. If, however, errors occur on two (physical) qubits QEC will be unable to restore the system to its proper state. Based on this we will show that QEC need not be appliled after every gate. Let us assume a perfectly encoded state and Clifford gates that can be implemented bit-wise but with probability $p \ll 1$ will (independently) cause an error on each qubit. The probability of an error on one qubit is then $7p - \mathcal{O}(p^2)$ and on two qubits is $21p^2 - \mathcal{O}(p^3)$. Thus, we can then be reasonably sure that at most only one qubit will have an error which will be corrected by QEC. Implementing two gates without applying QEC in between increases the probability that an error occurs to one qubit to $14p - \mathcal{O}(p^2)$ and that errors occur to two qubits to $84p^2 - \mathcal{O}(p^3)$. Nevertheless, the probability of errors on two qubits is still only second order in p and thus QEC applied after both gates will almost certainly correct the state of the system. When n gates are applied the probability of an error on one qubit is $7n - \mathcal{O}(p^2)$ and on two qubits $21(n + 2\binom{n}{2})p^2$. Still, the probability that two (or more) errors occur remains of order p^2 and QEC will correct the single qubit errors which will occur with probability of order p. From this we see that QEC is not needed until the end of the gate sequence since at no point will the probability of errors on two or more qubits be of order p (a similar argument can be made when including a two-qubit Clifford gate such as a controlled-NOT gate, this will be explored elsewhere).

If T-gates are included the implementation becomes more complex. However, assuming the T-gate is done following the rules of QFT, two qubit errors will still occur with probability of order p^2 and T-gates will thus behave like the Clfford gates described above: QEC need not be applied until the end of the gate sequence. Of course, if QEC could be implemented perfectly, and we were not concerned with resource consumption, it would be worthwhile to apply QEC as much as possible. This will lower even further the possibility of multiple errors. However, QEC cannot be done perfectly in any realistic system (and resource usage is a concern). Thus, we are left to ask, how often should QEC be applied? Applying noisy QEC too often will be expensive in terms of time and qubits and may in fact decrease the fidelity of the quantum information. However, not applying QEC often enough will allow error probabilities to grow so large that errors become likely.

To explore how often QEC should be applied we simulate single qubit gates appropriate for the [7,1,3] QEC code in a nonequiprobable Pauli operator error environment [21] with non-correlated errors. As in [22], this model is a stochastic version of a biased noise model that can be formulated in terms of Hamiltonians coupling the system to an environment. In the model used here, however, the probabilities with which the different error types take place is left arbitrary: the environment causes qubits to undergo a σ_x^j error with probability p_x, a σ_y^j error with probability p_y, and a σ_z^j error with probability p_z, where σ_i^j, $i = x, y, z$ are the Pauli spin operators on qubit j. We assume that only qubits taking part in a gate operation will be subject to error. Qubits not involved in a gate are assumed to be perfectly stored. While this represents an idealization, it is partially justified in that it is generally assumed that idle qubits are less likely to undergo error than those involved in gates (see for example [23]). In addition, in this paper accuracy measures are calculated only to second order in the error probabilities p_i thus the effect of ignoring storage errors is likely minimal. Finally, we note that non-equiprobable errors occur in the initialization of qubits to the $|0\rangle$ state and measurement (in the z or x bases) of all qubits.

To simulate the effects of less error correction we start with an arbitrary single qubit state, $|\psi\rangle = \cos\alpha|0\rangle + e^{i\beta}\sin\alpha|1\rangle$, perfectly encoded into the [7,1,3] error correction code. We then implement a series of gates, $...U_2U_1$, in the nonequiprobable error environment leading to a final state, ρ_f, of the 7 qubits. The final state is a function of the initial state, parameterized by α and β, and the error probabilities p_x, p_y, and p_z. We utilize two measures of accuracy comparing the simulated implementations with the final state after perfectly applied gates, ρ_i. The first is a state fidelity $\mathrm{Tr}[\rho_i\rho_f]$. The second is the logical gate fidelity, a state independent measure comparing the logical operation on the single qubit of encoded information to the ideal single qubit gate. To determine the logical gate fidelity we must first construct logical process matrices for the ideal and implemented operations. This is done by perfectly decoding ρ_f and tracing over all qubits except the first giving the logical single qubit output state. We then substitute α and β for the four specific states needed to calculate the process matrix as outlined in [1, 24]. The gate fidelity of the logical gate is then simply $\mathrm{Tr}[\chi_i\chi_f]$ where χ_i is the process matrix of the perfect gate and χ_f is the process matrix of the implemented logical gate.

After the gates, perfect (with no errors) or noisy (in the nonequiprobable error environment) QEC is applied to ρ_f giving final states ρ_{fp} and ρ_{fn} respectively. Based on our above argument we expect perfect QEC to affirm the 'correctability' of the errors that occur during implementation of multiple gates by raising the state or gate fidelity to unity (to at least second order in all p_i). In a realistic experiment, however, perfect QEC is not possible. Thus, we apply QEC to ρ_f in the nonequiprobable error environment in order to compare the application of QEC at different intervals in a more realistic scenario. To apply noisy QEC in a fault tolerant fashion we utilize four qubit ancilla Shor states [5] for syndrome measurement. The Shor states are themselves constructed in the nonequiprobable error environment and thus require verification. Based on the simulations of [25] we apply one verification step to each Shor state. Because every gate implemented in the nonequiprobable error environment has an error probability p_i the fidelity of ρ_{fn} will contain terms first order in p_i. Nevertheless, comparing ρ_{fn} for single and multiple gates will alert us if there is a significant decrease in fidelity due to lack of error correction after every gate.

We first look at gate sequences of Clifford gates only. Implementing a Clifford gate, C on the [7,1,3] QEC code requires implementing $C^\dagger$ on each of the 7 qubits. We simulate sequences of Clifford gates found interspersed between T-gates in typical approximations of arbitary rotations including H, PH and HPH, where H is the Hadamard gate and $P = T^2$ is a $\pi/2$ phase gate [11, 12, 16]. Results are shown in Table I up to first order in error probability (calculations were performed up to second order). Looking at both the state and logical gate fidelities for gate sequences with no error correction we see the expected steady decrease in fidelity as more gates are implemented. The decrease of the state fidelity is proportional to $7p_i$ the probability of single qubit errors, as discussed above. As expected, applying perfect error correction after one, two, or three Clifford gates gives state and logical gate fidelities of 1 (to third order).

Applying noisy QEC after the sequence of gates we find that the state and gate fidelities are exactly the same for one and two Clifford gates. This demonstrates that there is no need to apply QEC after only one gate. Noisy QEC applied after three gates gives a lower fidelity state than when applied after two gates. In both the two and three gate case, noisy QEC causes a decrease in

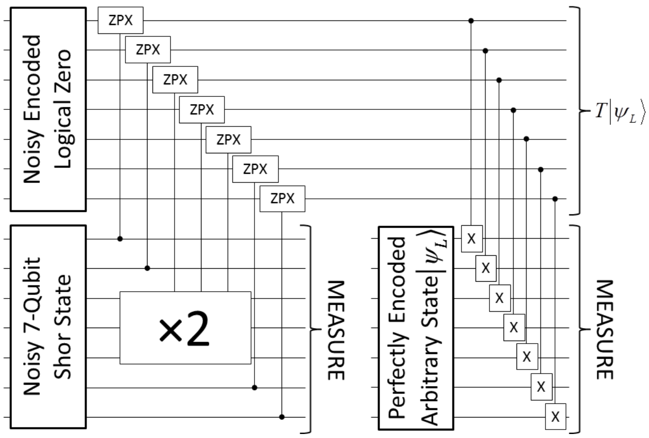

FIG. 1: Implementation of [7,1,3] QEC code T-gate.

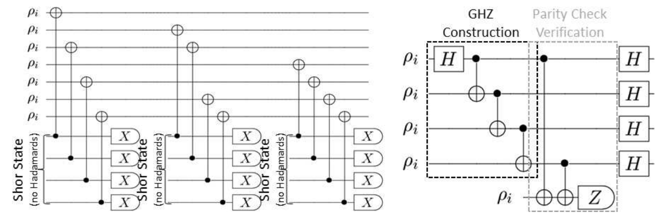

FIG. 2: Left: Circuit for phase syndrome measurement on the [7,1,3] QEC code, used here to initialize logical zero state. Right: Four qubit Shor state construction with one verification.

the state and gate fidelities when compared to the uncorrected state with respect to σ_x errors and an increase with respect to σ_z errors. Thus, if σ_x errors are dominant it is better to implement even more gates before applying QEC. The decrease with respect to σ_x errors can be attributed both to the fact that we have measured the bit-flip syndromes first (and thus additional, uncorrected σ_x errors occur during the phase-flip syndrome measurements), and to the use of noisy Shor states with one verification as demonstrated in [25]. The fidelity due to σ_y errors may increase or decrease upon application of noisy error correction.

The simulations of two or three Clifford gates followed by QEC should be compared to the case of applying QEC after each Clifford gate. The results of this latter simulation are shown in the last line of Table I. When the second gate is applied after the first application of QEC the fidelity is decreased with respect to p_x and p_y. The second application or QEC, however, increases the fidelity back to the same level as after the first QEC application. This implies that constant application of QEC will keep the fidelity steady. These simulations also underscore that there is no need to perform QEC after every gate as, after two gates, the gate and state fidelities are exactly the same whether or not QEC has been applied after the first gate.

We now look at sequences of two and three gates that include a T-gate. To implement a logical T-gate on a state encoded in the [7,1,3] QEC code we first construct the ancilla state $|\Theta\rangle = \frac{1}{\sqrt{2}}(|0_L\rangle + e^{i\frac{\pi}{4}}|1_L\rangle)$, where $|0_L\rangle$ and $|1_L\rangle$ are the logical basis states on the [7,1,3] QEC code. Bitwise CNOT gates are then applied between the state $|\Theta\rangle$ and the encoded state with the $|\Theta\rangle$ state qubits as control. Measurement of zero on the encoded state projects the encoded state with the application of a T-gate onto the qubits that had made up the $|\Theta\rangle$ state.

To ensure fault tolerance in the construction of $|\Theta\rangle$ requires the following steps: (1) A logical zero state is encoded by applying error correction to 7 qubits all initially in the state $|0\rangle$ [4]. We choose to use Shor state ancilla for syndrome measurements [5] each of which undergoes one verification step [25]. (2) A seven qubit Shor state [5] in constructed and proper verifications are applied. (3) Seven controlled-ZPX gates, given by:

$$C - ZPX \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{i\frac{\pi}{4}} \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \end{pmatrix}, \qquad (1)$$

are applied each between a qubit of the Shor state and a qubit of the logical zero state with the Shor state qubits as control. (4) Measurement of the Shor state (with even parity outcome) completes the projection and the construction of the logical state $|\Theta\rangle$. Circuits for these steps are shown in Figs. 1 and 2.

For our simulations, done in the nonequiprobable error environment, the following should be noted [20]. First, each attempted initialization of the state $|0\rangle$, for the logical zero state and Shor states, instead initializes to the state $\rho_i = (1 - p_x - p_y)|0\rangle\langle 0| + (p_x + p_y)|1\rangle\langle 1|$. Second, were initialization perfect there would be no need to perform the bit-flip syndrome measurements when encoding the logical zero state. Here, though initialization is not perfect, we apply only the phase-flip syndrome measurements (each syndrome measurement is performed twice to conform with the strictures of fault tolerance). Third, because logical zero state encoding is done ‘off-line’ we choose to post-select only the encoded states where all syndrome measurements are zero. Finally, the projection into the state $|\Theta\rangle$ is done until the same measurement outcome is attained twice in a row to ensure no errors have taken place during the projection itself.

Our simulations of a T-gate, when implemented in the fault tolerant way described above, work as expected. Namely, when followed by perfect QEC the gate fidelity is unity up to second order [20]. In Table II we compare the simulation of the T-gate alone with that of a T-gate with one, two, or three Clifford gates. Once again, the question we are trying to address is how often noisy QEC should be applied.

The first point of interest is the difference in fideli-

TABLE I: Fidelity measures of Clifford gates implemented in the nonequiprobable error environment with and without noisy error correction applied. We define $s_1 = \cos(4\alpha)$ and $s_2 = \cos(2\beta)\sin(2\alpha)^2$.

State Fidelity	no QEC	noisy QEC	Gate Fidelity	no QEC	noisy QEC
H or P	$1 - 7p_x - 7p_y - 7p_z$	$1 - 73p_x - 19p_y - 7p_z$	H or P	$1 - 3p_x - 5p_y - 3p_z$	$1 - 19p_x - 5p_y - 3p_z$
PH	$1 - 14p_x - 14p_y - 14p_z$	$1 - 73p_x - 19p_y - 7p_z$	PH	$1 - 8p_x - 8p_y - 6p_z$	$1 - 19p_x - 5p_y - 3p_z$
HPH	$1 - 21p_x - 21p_y - 21p_z$	$1 - \frac{(155-s_1-2s_2)p_x}{2} - \frac{(97-3s_1-6s_2)p_y}{4} - \frac{(61-3s_1-6s_2)p_z}{4}$	HPH	$1 - 11p_x - 13p_y - 9p_z$	$1 - 23p_x - 11p_y - 8p_z$
P-QEC-H	$1 - 80p_x - 26p_y - 14p_z$	$1 - 73p_x - 19p_y - 7p_z$	P-QEC-H	$1 - 54p_x - 15p_y - 6p_z$	$1 - 19p_x - 5p_y - 3p_z$

TABLE II: Fidelity measures of Clifford gates implemented in the nonequiprobable error environment with and without noisy error correction applied.

State Fidelity	no QEC	noisy QEC	Gate Fidelity	no QEC	noisy QEC
T	$1 - 7p_x - 7p_y - 26p_z$	$1 - 73p_x - 19p_y - 7p_z$	T	$1 - 3p_x - 5p_y - 14p_z$	$1 - 19p_x - 5p_y - 3p_z$
PT	$1 - 14p_x - 14p_y - 33p_z$	$1 - 73p_x - 19p_y - 7p_z$	PT	$1 - 8p_x - 8p_y - 17p_z$	$1 - 19p_x - 5p_y - 3p_z$
HT	$1 - 14p_x - 14p_y - 33p_z$	$1 - 73p_x - 19p_y - 7p_z$	HT	$1 - 6p_x - 10p_y - 17p_z$	$1 - 19p_x - 5p_y - 3p_z$
TPH	$1 - 7p_x - 7p_y - 40p_z$	$1 - 73p_x - 19p_y - 7p_z$	TPH	$1 - 3p_x - 5p_y - 20p_z$	$1 - 19p_x - 5p_y - 3p_z$
THPH	$1 - 14p_x - 14p_y - 33p_z$	$1 - 73p_x - 19p_y - 7p_z$	THPH	$1 - 6p_x - 8p_y - 17p_z$	$1 - 19p_x - 5p_y - 3p_z$
P-QEC-T	$1 - 73p_x - 19p_y - 7p_z$	$1 - 73p_x - 19p_y - 7p_z$	P-QEC-T	$1 - 19p_x - 5p_y - 3p_z$	$1 - 19p_x - 5p_y - 3p_z$

ties between the T-gate and that of the Clifford gates P and H. When looking at the implementation of single gates without error correction we see that the fidelity as a function of p_x and p_y are the same. The T-gate, however, has significantly increased sensitivity to σ_z errors. In other words, the accuracy 'cost' (there is, of course, a prohibitive cost in the number of extra qubits utilized and in the time of implementation) of applying a T-gate as opposed to a single qubit Clifford gate is only with respect to phase errors. Applying noisy QEC to the single gates gives equal fidelities for the T-gate and Clifford gates. This is likely because the first order error terms arising from the implementation of the gate are corrected by QEC and thus the remaining first order error terms are due to the QEC itself.

Implementing a single Clifford gate after a T-gate decreases the fidelity by the same amount as applying a Clifford gate after another Clifford gate. Implementing a T-gate after two Clifford gates decreases the fidelity compared to the T-gate alone only with respect to p_z and, in fact, the fidelity with respect to p_x and p_y is higher than that of two Clifford gates alone. Implementing the T gate after three Clifford gates gives fidelity equal to applying a Clifford gate after the T gate. Thus, the simulations show some complexity in terms of which gates will decrease or increase the fidelity with respect to the different error types.

Applying perfect QEC after any of the above gate sequences gives unit fidelity to second order in p_i (as opposed to third order for the Clifford only gate sequences). As expected, the errors are correctable and QEC need not be applied at any additional point during the sequence.

We mentioned above that the fidelity measures after noisy QEC appear to be insensitive to the gates applied before QEC. This is clearly seen in Table II. Presumably this arises because the QEC corrects the errors of the previous gates or at least increases their order in error probability (in line with the perfect error correction simulations), and the noise inherent in the QEC is solely responsible for the first order error terms.

These simulations should also be compared to the case of applying QEC after an intial T-gate and then again after a Clifford gate. The results of this latter simulation with the P gate are shown in the last line of Table II. Implementation of P after the initial QEC does not change the fidelity and the fidelity remains constant upon the second application of QEC. This implies that constant application of QEC will keep the fidelity steady and again underscores that there is no need to perform QEC after every gate.

When implementing an arbitrary rotation from the gate set Clifford plus T one must trade off accuracy, ϵ, versus number of gates. The more gates used the better the accuracy but the higher cost in qubits and time. The need to apply (noisy) QEC bounds the accuracy with which the arbitrary rotation can be implemented. Therefore, it may not be worth spending resources to achieve highly accurate rotations if the accuracy will be destroyed by the application of noisy QEC. To determine how accurate the gate sequence should be and how often QEC should be applied will require futher detailed simulations.

In conclusion, we have explored the question of how often quantum error correction needs to applied during a sequence of logical single qubit gates from the gate set Clifford plus T as would be necessary for the implementation of arbitrary single qubit rotations. We have shown why QEC is actually necessary only at the end of such a sequence and demonstrated that for practical implementations in which QEC is imperfect, there is likely to be a loss in fidelity if QEC is applied too often. All of our simulations were done within the [7,1,3] QEC code but the results should be directly applicable to other CSS codes and, perhaps, to other QEC codes as well.

In addition, we have utilized logical χ-matrices in evaluating the logical gate fidelity of the Clifford and T-gates. The χ-matrix is easily transformed into Kraus operators which properly describe the one-qubit sequence: perfect encoding into the [7,1,3] code, implementation of gate, possible implementation of QEC, perfect decoding. Such Kraus operators may be useful for simulations of quantum fault tolerance.

I would like to thank G. Gilbert for insightful comments and S. Buchbinder for help on the initial stages of this work. This research is supported under MITRE Innovation Program Grant 51MSR662. Approved for Public Release; Distribution unlimited. 13-1885.

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