B.W. Lamar, "Min-Additive Utility Functions," pp. 1–32.

Min-Additive Utility Functions

by

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February 2009

ABSTRACT

This paper introduces the "min-additive" (also called "min-average") utility function. This function is a weighted combination of an additive utility function and a minimization over a set of single attribute utility functions. The weighting is accomplished by exploiting information already contained in the additive and minimization models. Four forms of the min-additive (MA) model are presented—basic, uniform, logistic, and relaxed. The basic MA model generalizes the additive and minimization models but does not require any additional parameters to be estimated. It can be employed in situations where the decision-maker's preferences violate the additive independence assumptions inherent in the additive model. The uniform MA model extends the basic MA model by adding "location" and "spread" parameters. The logistic MA model extends the uniform MA model by creating a continuously differentiable weighting function. This weighting function is shown to be a close approximation of a Gaussian cumulative distribution function. The relaxed MA model removes the non-negativity requirements on the weights. This version of the MA model is shown to be a generalization of the two-dimensional multi-linear utility function (and the two-dimensional multiplicative utility function). Numerical examples and graphical representations of the models are presented. The paper contains three appendices. Appendix A illustrates how the MA model can be nested in a decision preference hierarchy. Appendix B compares the MA model to the recently proposed "limited average" and "exponential-average" family of utility functions. Finally, Appendix C summarizes the complement to the MA model—the max-additive model. The max-additive model is used in risk analysis and other situations involving disutilities.

1. Introduction

Let $\underline{\boldsymbol{x}} = \{x_1, x_2, \ldots, x_n\}$ be the set of attributes that are of interest to the decision maker (DM). Each attribute x_i varies from a least preferred value x_i^{WORST} to a most preferred value x_i^{BEST}. A *utility function* [Keeney (1992), p.132] is a real-valued function that expresses the DM's strength of preference for various levels of $\underline{\boldsymbol{x}}$. In practice, the most commonly used utility function is the *additive model* of the form

$$\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})) = \sum_{i=1}^{i=n} w_i \cdot u_i(x_i) \qquad (1)$$

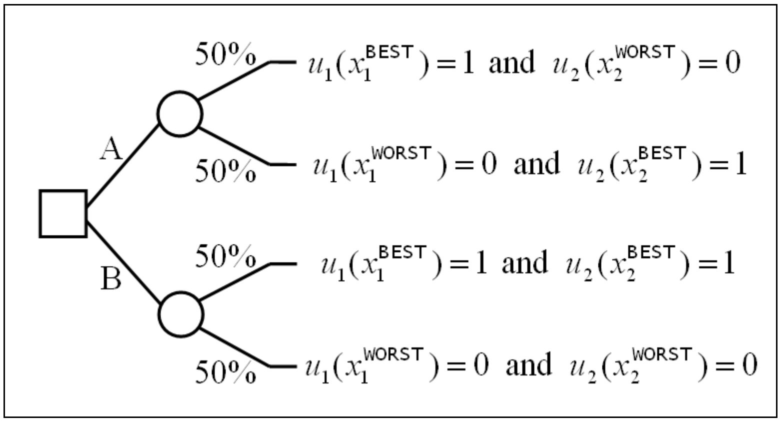


Figure 1–Alternatives A and B

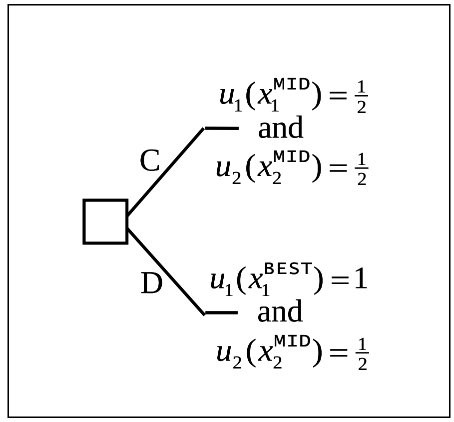


Figure 2–Alternatives C and D

where the w_i 's are a set of *n* non-negative weights (i.e., constants) that sum to unity, the $u_i(x_i)$'s are a set of *n* single dimensional (i.e., single attribute) utility functions and $\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})$ = $\{u_1(x_1), u_2(x_2), \ldots, u_n(x_n)\}$. Typically, each $u_i(x_i)$ is calibrated so that $u_i(x_i^{\text{WORST}}) = 0$ and $u_i(x_i^{\text{BEST}}) = 1$. Then $\text{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ also ranges between 0 and 1. There is an extensive and well-established literature for determining the functional form of the $u_i(x_i)$'s and for assessing the values of the w_i 's in the additive model (see, for example, [Clemen and Reilly (2004)], [Keeney and Raiffa (1976)], [Kirkwood (1997)], [Mollaghasemi and Pet-Edwards (1997)], [Pomerol and Barba-Romero (2000)], [Raiffa (1968)], and [von Winterfeldt and Edwards (1986)]).

The additive model assumes *additive independence* [Fishburn (1964), pp. 43–47], [Keeney and Raiffa (1976), pp. 295–297] meaning, simply, that $\text{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ is assumed to be a linear function with respect to the individual utilities, u_i. Thus, $\partial\text{ADD} / \partial u_i$, the marginal rate of change of $\text{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ with respect to u_i, is assumed to be constant (namely the weight w_i) regardless of the value of *any* of the individual utilities $u_1, u_2, \ldots, u_n$. [Clemen and Reilly (2004), p. 585], [Kirkwood (1997), p. 250], [von Winterfeldt and Edwards (1986), p. 309], and others point out that additive independence is, in some cases, an unrealistic assumption. A classic example (see, for example, [Kirkwood (1997), p. 250]) illustrating the implications of this assumption involves a project manager (the DM) who is considering two attributes—project cost (x_1) and project schedule (x_2). The cost attribute can vary from x_1^{BEST}(project completed in budget) to x_1^{WORST}(project suffers unacceptable cost overruns). The schedule attribute can vary from x_2^{BEST}(project completed on schedule) to x_2^{WORST}(project suffers unacceptable delays). We set $u_1(x_1^{\text{WORST}}) = 0$, $u_1(x_1^{\text{BEST}}) = 1$, $u_2(x_2^{\text{WORST}}) = 0$, and $u_2(x_2^{\text{BEST}}) = 1$. The DM is asked to consider two alternatives—A and B (see Figure 1). For alternative A there is a 50% chance that the project will be in budget but will fail due to unacceptable schedule delays, and a 50% chance that the project will be on schedule but will fail due to unacceptable cost overruns). In contrast, for alternative B, there is 50% chance that the project will be completed in budget and on time, and a 50% chance that the project will have both unacceptable cost overruns and unacceptable delays.

As shown in eq. (2), under the additive independence assumption, the expected utility for the two alternatives is the same implying that the DM is indifferent between the two alternatives:

$$\begin{aligned} \mathrm{E}[\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{A}] &= \tfrac{1}{2}\cdot(w_1\cdot 1 + w_2\cdot 0) + \tfrac{1}{2}\cdot(w_1\cdot 0 + w_2\cdot 1) = \tfrac{1}{2} \\ \mathrm{E}[\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{B}] &= \tfrac{1}{2}\cdot(w_1\cdot 1 + w_2\cdot 1) + \tfrac{1}{2}\cdot(w_1\cdot 0 + w_2\cdot 0) = \tfrac{1}{2} \end{aligned} \tag{2}$$

However, this indifference conclusion is unrealistic since it assumes that the DM is willing to tradeoff unacceptable outcomes in the same manner as she/he is willing to tradeoff acceptable outcomes. In reality, if the project is failing because of unacceptable cost overruns, the DM is more likely to have a low utility assessment regardless in the status to the schedule attribute. Similarly, if the project is experiencing unacceptable schedule delays, the DM will have a low utility assessment irrespective of the project costs. Thus, when unacceptable outcomes are being considered the DM's strength of preference is more accurately represented by a *minimization model* of the form

$$\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) = \min\{u_1(x_1), u_2(x_2), \ldots, u_n(x_n)\} \tag{3}$$

where, as with eq. (1), the $u_i(x_i)$'s are the set of *n* single dimensional utility functions. Applying the minimization model to the two alternatives depicted in Figure 1 yields

$$\begin{aligned} \mathrm{E}[\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{A}] &= \tfrac{1}{2}\cdot\min\{1,0\} + \tfrac{1}{2}\cdot\min\{0,1\} = 0 \\ \mathrm{E}[\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{B}] &= \tfrac{1}{2}\cdot\min\{1,1\} + \tfrac{1}{2}\cdot\min\{0,0\} = \tfrac{1}{2} \end{aligned} \tag{4}$$

The minimization model indicates that the DM prefers alternative B (with a 50% chance of project success) to alternative A (with a 0% chance of project success).

This example was specifically constructed to illustrate the limitations of the additive model. But the minimization model has limitations as well. To illustrate the shortcomings of the minimization model, suppose the DM (i.e., the project manager) is asked to consider another pair of alternatives—C and D (see Figure 2). Let $x_1^{\mathtt{MID}}$ be the value of x_1 such that $u_1(x_1^{\mathtt{MID}}) = \frac{1}{2}$; similarly, let $x_2^{\mathtt{MID}}$ be the value of x_2 such that $u_2(x_2^{\mathtt{MID}}) = \frac{1}{2}$. In alternative C, $x_1 = x_1^{\mathtt{MID}}$ and $x_2 = x_2^{\mathtt{MID}}$ so that $u_1(x_1) = \frac{1}{2}$ and $u_2(x_2) = \frac{1}{2}$. On the other hand, in alternative D, $x_1 = x_1^{\mathtt{BEST}}$ and $x_2 = x_2^{\mathtt{MID}}$ so that $u_1(x_1) = 1$ and $u_2(x_2) = \frac{1}{2}$. Note that alternative D is at least as good as alternative C in all the attributes and strictly better than alternative C in at least one attribute. Thus, alternative D dominates alternative C, implying that the DM should prefer alternative D to alternative C. The DM's preference for alternative D is reflected in the additive model which indicates that the DM would prefer alternative D as long as $w_1 = \partial \mathsf{ADD} / \partial u_1 > 0$:

$$\begin{aligned} \mathrm{E}[\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{C}] &= w_1\cdot\tfrac{1}{2} + w_2\cdot\tfrac{1}{2} = \tfrac{1}{2} \\ \mathrm{E}[\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{D}] &= w_1\cdot 1 + w_2\cdot\tfrac{1}{2} = \tfrac{1}{2} + \tfrac{1}{2}\cdot w_1 \end{aligned} \tag{5}$$

In contrast, the minimization model implies that the DM would be indifferent between alternatives C and D:

$$\begin{aligned} \mathrm{E}[\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{C}] &= \min\left\{\tfrac{1}{2}, \tfrac{1}{2}\right\} = \tfrac{1}{2} \\ \mathrm{E}[\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{D}] &= \min\left\{1, \tfrac{1}{2}\right\} = \tfrac{1}{2} \end{aligned} \tag{6}$$

This discussion motivates the need for a composite utility function that captures the desirable aspects of the additive and minimization models while, at the same time, avoids the pitfalls inherent in each of these two models.

One established approach for incorporating the properties of both the additive and the minimization model is the *multi-linear model* [Keeney and Raiffa (1976), pp. 293–294]. This model extends the additive model to include the cross-products of the individual utility functions. The multi-linear model is of the form

$$\begin{aligned} \mathtt{MULTILIN}(\underline{\boldsymbol{u}}(\underline{x})) &= \sum_{i=1}^{i=n} w_i \cdot u_i(x_i) + \sum_{i=1}^{i=n-1} \sum_{j=i+1}^{j=n} w_{ij} \cdot u_i(x_i) \cdot u_j(x_j) \\ &+ \sum_{i=1}^{i=n-2} \sum_{j=i+1}^{j=n-1} \sum_{k=j+1}^{j=n} w_{ijk} \cdot u_i(x_i) \cdot u_j(x_j) \cdot u_k(x_k) + \quad \cdots \quad + \prod_{i=1}^{n} w_{1,\ldots,n} \cdot u_i(x_i) \end{aligned} \tag{7}$$

where the w_i's, w_{ij}'s, w_{ijk}'s,…, $w_{1,\ldots,n}$ are a set of weights (i.e., constants) and the $u_i(x_i)$'s are a set of *n* single dimensional utility functions. The first summation in eq. (7) is the additive model. The remaining terms in this equation are cross products of the individual utility functions. If any of the $u_i(x_i)$'s in a cross-product term is zero, then the value of that term is zero. Thus, for unacceptable outcomes, the cross-product terms mirror the effect of the minimization model. Although the multi-linear model has been successfully applied in practice, it has two drawbacks. First, note that the multi-linear model contains $2^n - 1$ weights (w_i, w_{ij}, etc.). Thus, the number of weights that need to be assessed skyrockets as the number of individual utility functions increases. Second, in many cases the cross-product terms do not have a meaningful interpretation to the DM. Other extensions of the additive model, such as the multiplicative model and higher order polynomials, have been proposed [Krantz et. al (1971), pp. 321–328]. However, they suffer from some of the same drawbacks as the multi-linear model.

In this paper, we propose an alternative approach for melding the characteristics of the additive and minimization models. This approach combines the additive and minimization forms in a straightforward manner, requires few additional parameters to be estimated, can be visualized graphically, and has a relatively easy interpretation for the DM (and the analyst!). We call this form of utility function the *min-additive model*.[1]

The remainder of this paper is organized as follows. Section 2 introduces the basic min-additive model. Section 3 extends the basic MA model by adding "location" and "spread" parameters using a uniformly distributed weighting function. Section 4 offers a further refinement of the basic MA model using a logistic weighting function. This weighting function closely approximates a Gaussian cumulative distribution function. Section 5 proves that a variation of the uniform MA model is a generalization of the two-dimensional multi-linear model (and the two-dimensional multiplicative utility function). Section 6 gives a graphical representation of the min-additive model and provides a numerical example. Section 7 summarizes the paper. Appendix A illustrates how the min-additive model can be nested in a decision-making hierarchy. Appendix B compares the min-additive model with the recently proposed "limited average" [Moynihan and Shimi (2004)] and the "exponential average" [Schmidt (2007)] models. Appendix C describes the complement to the MA model—the max-additive model—used to represent the DM's preference structure in situations involving disutilities.

2. Basic Min-Additive Model

Note that both the additive and the multi-linear models use constant coefficients (i.e., the weights). However, if the weights to be assessed in a utility function are permitted to be functions of the attributes (rather than constants), then the distinction between a weight and a utility blurs. They both are functions of the attributes, they both vary between 0 and 1, and they both express the preferences of the DM. Thus, permitting the weights to be functions actually eases the burden of developing a utility function since the utilities themselves can serve as weights. In particular, consider a weighted combination of the additive and minimization models of the form

$$\mathsf{MA}(\underline{\boldsymbol{u}}(\underline{x})) = w_{\mathsf{MIN}}(\underline{\boldsymbol{u}}(\underline{x}))\cdot\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) + w_{\mathsf{ADD}}(\underline{\boldsymbol{u}}(\underline{x}))\cdot\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \tag{8}$$

where $\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ and $\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ are defined in eqs.(1) and (3), respectively, and $w_{\mathsf{ADD}}(\underline{\boldsymbol{u}}(\underline{x}))$ and $w_{\mathsf{MIN}}(\underline{\boldsymbol{u}}(\underline{x}))$ are non-negative weight *functions* that sum to one. The next question is: What should be the functional form of the weights $w_{\mathsf{ADD}}(\underline{\boldsymbol{u}}(\underline{x}))$ and $w_{\mathsf{MIN}}(\underline{\boldsymbol{u}}(\underline{x}))$? But, in principal, we have already answered this question. For instance, as we observed in the Introduction, the importance of the $\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ model increases as the minimum value of the individual utility functions (i.e., the $u_i(x_i)$'s) approaches zero. But, the minimum value of the individual utility functions is simply $\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ itself. Thus, the information about the value of the weight of $\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ can be inferred directly from $\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$, or more precisely, from the complement of

[1] In previous work, we used the label "min-average model." [Lamar and Schmidt (2004)]. Thus, "min-additive" and "min-average" are synonyms for the same model.

$\mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$. In other words, $w_{\mathsf{MIN}}(\underline{\boldsymbol{u}}(\underline{x})) = 1 - \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$. Furthermore, since the weights in eq. (8) sum to one, we have $w_{\mathsf{ADD}}(\underline{\boldsymbol{u}}(\underline{x})) = \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$. Thus, the *basic form* of the *min-additive model* is

$$\mathsf{MA}_0(\underline{\boldsymbol{u}}(\underline{x})) = \left(1 - \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))\right) \cdot \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \;+\; \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \cdot \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \tag{9}$$

where $\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ is defined in eq. (1) and $\mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ is defined in eq.(3). The subscript "0" attached to $\mathsf{MA}_0(\underline{\boldsymbol{u}}(\underline{x}))$ is used to denote the "basic form" of the min-additive model. Three extensions of the model—denoted $\mathsf{MA}_1(\underline{\boldsymbol{u}}(\underline{x}))$, $\mathsf{MA}_2(\underline{\boldsymbol{u}}(\underline{x}))$ and $\mathsf{MA}_3(\underline{\boldsymbol{u}}(\underline{x}))$—are presented in Sections 3 through 5.

To illustrate the performance of the basic min-additive model, we apply this model to the four alternatives—A, B, C, and D—depicted in Figures 1 and 2.

$$\begin{aligned}
&\mathrm{E}[\mathsf{MA}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{A}] = \tfrac{1}{2} \cdot \left((1-0) \cdot 0 + 0 \cdot w_1\right) + \tfrac{1}{2} \cdot \left((1-0) \cdot 0 + 0 \cdot w_2\right) = 0 \\
&\mathrm{E}[\mathsf{MA}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{B}] = \tfrac{1}{2} \cdot \left((1-1) \cdot 1 + 1 \cdot 1\right) + \tfrac{1}{2} \cdot \left((1-0) \cdot 0 + 0 \cdot 0\right) = \tfrac{1}{2} \\
&\mathrm{E}[\mathsf{MA}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{C}] = (1 - \tfrac{1}{2}) \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{2} \\
&\mathrm{E}[\mathsf{MA}(\underline{\boldsymbol{u}}(\underline{x})) \mid \mathrm{D}] = (1 - \tfrac{1}{2}) \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot (w_1 \cdot 1 + w_2 \cdot \tfrac{1}{2}) = \tfrac{1}{2} + \tfrac{1}{4} \cdot w_1
\end{aligned} \tag{10}$$

The calculations in eq. (10) show that, according to the basic min-additive model, alternative B is preferred to alternative A. This preference is reasonable since alternative B has a 50% chance of project success, whereas alternative A has a 0% chance of project success. The min-additive model also indicates that the DM is indifferent between alternatives B and C. In addition, the min-additive model indicates that alternative D is preferred to alternative C. This preference is reasonable since alternative D dominates alternative C.

In comparing eq. (10) with eq. (5), we see that the utility value of alternative D is lower in the min-additive model than it is in the additive model ($\tfrac{1}{2} + \tfrac{1}{4} \cdot w_1$ compared to $\tfrac{1}{2} + \tfrac{1}{2} \cdot w_1$). The min-additive utility value is lower because this model "discounts" the utility value of an alternative based on the proximity of the attributes associated with that alternative. As the attribute values get closer and closer to an unacceptable value, the "discounting" of the utility value gets greater and greater. Thus, from a mathematical programming perspective, the min-additive model can be viewed as a convex combination of a utility maximizing objective function and a barrier function [Bazaraa and Shetty (1979), pp. 342–349] constraining the solution space to a feasible region. This viewpoint also enables us to refine the definition of x_i^{WORST} such that $u_i(x_i^{\mathsf{WORST}}) = 0$.

Let x_i^{WORST} be the value of attribute x_i beyond which the solution is infeasible to the DM.

In the project management example given above, x_1^{WORST} was defined as an unacceptable cost overrun and x_2^{WORST} was defined as an unacceptable schedule delay. That example implied dire consequences if the value of x_i^{WORST} was exceeded. However, that need not be the case. For example, [von Winterfeldt and Edwards (1986), p. 236] describe the facility location problem of an executive (the DM) in a consulting firm who wants to open a branch office in southern California. One attribute of DM's location decision is the distance of the office from LAX international airport. In this example, the DM is unwilling to consider sites that are more than one-hour's driving time from LAX[2]. Hence, $x_i^{\mathsf{WORST}} = 60$ minutes for the "driving time" attribute, x_i. There is nothing wrong, per say, with offices located more than an hour away from LAX. They are just not feasible solutions to the DM's decision problem at hand.

The next three sections present three extensions of the basic min-additive model.

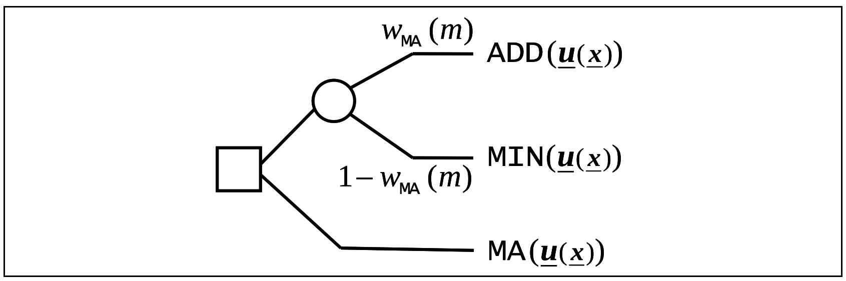


Figure 3–Probability interpretation of MA weight function

3. Uniform Min-Additive Model

Let $m = \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ and let $w_{\mathsf{MA}}(m)$ be the weight applied to the $\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ term in eq. (8). We call $w_{\mathsf{MA}}(m)$ the "*min-additive weight function*" (or "*MA weight function*," for short). By substituting the MA weight into eq. (8), this equation can be re-written as

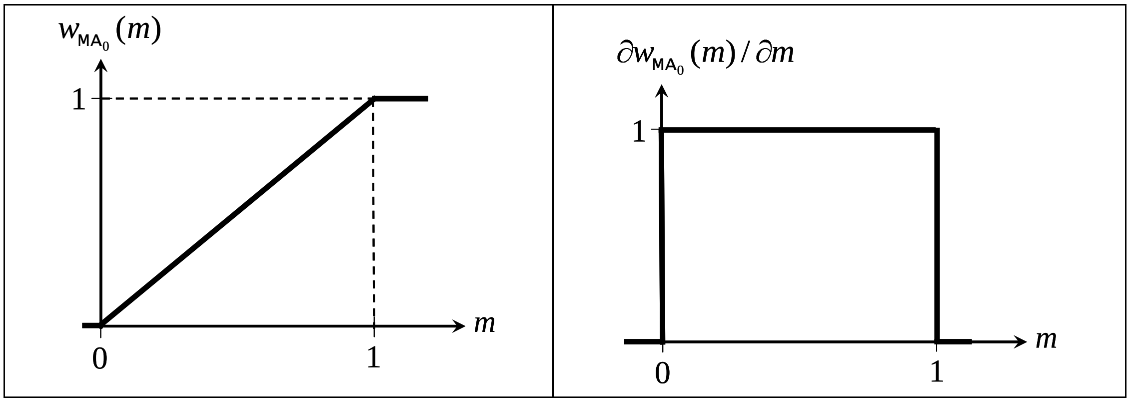
$$\mathsf{MA}(\underline{\boldsymbol{u}}(\underline{x})) = \left(1 - w_{\mathsf{MA}}(m)\right) \cdot \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \; + \; w_{\mathsf{MA}}(m) \cdot \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \tag{11}$$

For any given m, $w_{\mathsf{MA}}(m)$ can be interpreted as the (conditional) probability that the DM prefers the additive model rather than the minimization model; and the value of the $\mathsf{MA}(\underline{\boldsymbol{u}}(\underline{x}))$ utility function can be interpreted as the conditional expected value of these two models (see Figure 3). Let $w_{\mathsf{MA0}}(m)$ denote the value of the MA weight, $w_{\mathsf{MA}}(m)$, in the basic min-additive model given in eq. (9). The value of $w_{\mathsf{MA0}}(m)$ increases uniformly from 0 to 1 as m varies from 0 to 1. That is,

$$w_{\mathsf{MA}_0}(m) = \begin{cases} 0 & \text{if } \; m < 0 \\ m & \text{if } \; 0 \le m \le 1 \\ 1 & \text{if } \; m > 0 \end{cases} \tag{12}$$

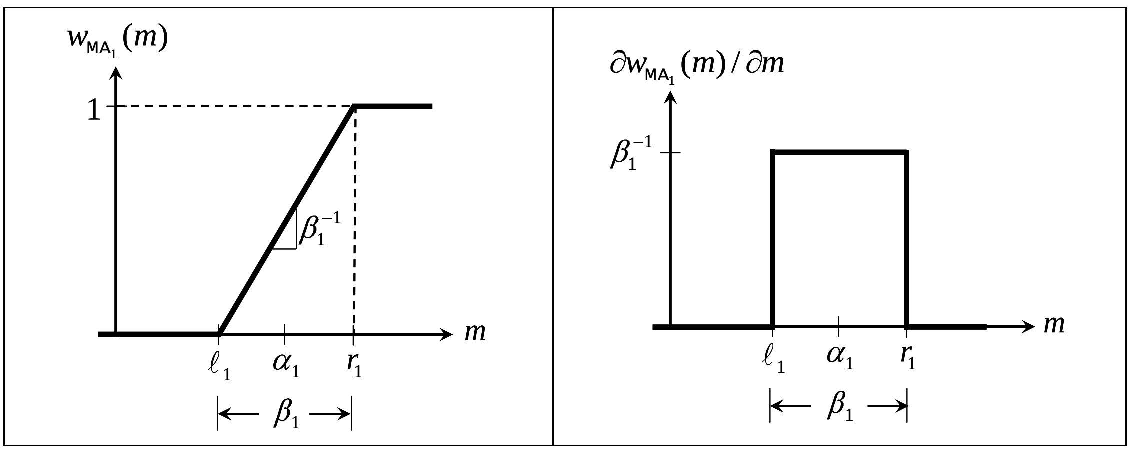
In fact, $w_{\mathsf{MA0}}(m)$ is the cumulative distribution function (CDF) of a uniformly distributed random variable in the domain [0,1] (see Figure 4).

[2] Approximately 2 miles during rush-hour.



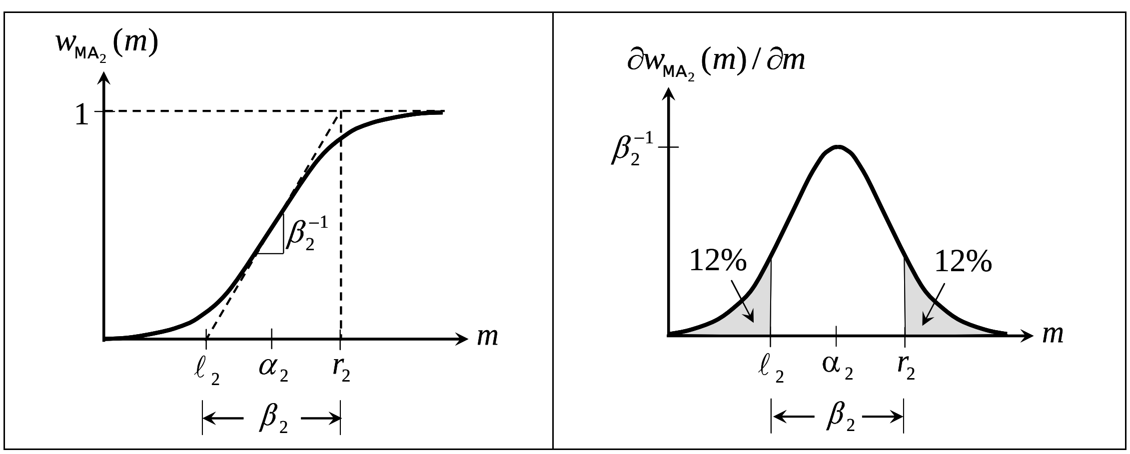
(a) Weight function (b) Derivative of weight function

Figure 4–Basic MA weight function



(a) Weight function (b) Derivative of weight function

Figure 5–Uniform MA weight function



(a) Weight function (b) Derivative of weight function

Figure 6–Logistic MA weight function

In order to extend the basic min-additive model, let ℓ_1 be the value of *m* below which the DM exclusively accepts the minimization model; and let r_1 be the value of *m* above which the DM exclusively accepts the additive model. In the basic min-additive model, the left endpoint, ℓ_1, is fixed at 0 and the right endpoint, r_1, is fixed at 1. A natural extension of the basic min-additive model is to allow ℓ_1 and r_1 to be parameters. We call this extension the *uniform min-additive model* whose utility function value is denoted by $\mathsf{MA}_1(\underline{\boldsymbol{u}}(\underline{x}))$ and whose MA weight function is denoted by $w_{\mathsf{MA}_1}(m)$. In the uniform min-additive model, $w_{\mathsf{MA}_1}(m)$ is the CDF of a uniformly distributed random variable in the domain $[\ell_1, r_1]$ (see Figure 5). Specifically,

$$\mathsf{MA}_1(\underline{\boldsymbol{u}}(\underline{x})) = \left(1 - w_{\mathsf{MA}_1}(m)\right) \cdot \mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \;+\; w_{\mathsf{MA}_1}(m) \cdot \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \tag{13}$$

where $\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ and $\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ are defined in eqs.(1) and (3), respectively, and $w_{\mathsf{MA}_1}(m)$ is given by

$$w_{\mathsf{MA}_1}(m) = \begin{cases} 0 & \text{if } m < \alpha_1 - \tfrac{1}{2} \cdot \beta_1 \\ \tfrac{1}{2} + \dfrac{m - \alpha_1}{\beta_1} & \text{if } \alpha_1 - \tfrac{1}{2} \cdot \beta_1 \le m \le \alpha_1 + \tfrac{1}{2} \cdot \beta_1 \\ 1 & \text{if } m > \alpha_1 + \tfrac{1}{2} \cdot \beta_1 \end{cases} \tag{14}$$

where α_1 is the mid-point between ℓ_1 and r_1, and β_1 is the difference between ℓ_1 and r_1 (see Figure 5). That is,

$$\alpha_1 = \tfrac{1}{2} \cdot (r_1 + \ell_1) \qquad \text{and} \qquad \beta_1 = r_1 - \ell_1 \tag{15}$$

Inverting eq. (15) gives

$$\ell_1 = \alpha_1 - \tfrac{1}{2} \cdot \beta_1 \qquad \text{and} \qquad r_1 = \alpha_1 + \tfrac{1}{2} \cdot \beta_1 \tag{16}$$

Note that α_1 is a "measure of location" and β_1 is a "measure of spread" of the distribution. Thus, varying α_1 shifts the distribution to the left or right along the real number line; and varying β_1 alters the distance between the left and right endpoints, ℓ_1 and r_1. Equivalently, β_1^{-1} (i.e., the reciprocal of β_1) is equal to the maximum rate of change of $w_{\mathsf{MA}_1}(m)$ with respect to *m*, and thus β_1^{-1} is a "measure of steepness" of the distribution.

These parameters can be set to represent the preferences of the DM. For example, if $\alpha_1 = 0.5$ and $\beta_1 = 0.4$, using eq. (16) gives $\ell_1 = 0.3$ and $r_1 = 0.7$. These parameter values indicate that the DM strictly prefers the minimization model (eq. (3)) if any of the single dimensional utility function (i.e., the $u_i(x_i)$'s) has a value below 0.3. Conversely, the DM strictly prefers the additive model (eq.(1)) if all of the $u_i(x_i)$ values are above 0.7. Otherwise, the DM prefers a weighted combination of the additive model and the minimization model (with the weight given by eq. (14)).

Note that the parameters ℓ_1 and r_1 are *not* required to be between 0 and 1. The only restriction placed on these two parameters is that $\ell_1 \neq r_1$. This gives a fair amount of flexibility in calibrating the uniform min-additive model to reflect the DM's preferences. In particular,

$$\text{If } \ell_1 < r_1 \text{ and } r_1 \leq 0, \quad \text{then} \quad \mathsf{MA}_1(\underline{\boldsymbol{u}}(\underline{x})) = \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \tag{17a}$$

$$\text{If } \ell_1 \geq 1 \text{ and } r_1 > \ell_1, \quad \text{then} \quad \mathsf{MA}_1(\underline{\boldsymbol{u}}(\underline{x})) = \mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \tag{17b}$$

The next section presents a further extension of the basic min-additive model.

4. Logistic Min-Additive Model

The uniform min-additive model discussed in the previous section uses a piecewise-linear MA weight function (see Figure 5 and eq. (14)). As noted, the parameters ℓ_1 and r_1 (or correspondingly α_1 and β_1) can be adjusted to reflect a range of DM preferences. But, despite these advantages, the piecewise-linear functional form also has two drawbacks—one practical and the other mathematical. First, as a practical matter, the DM may be uncomfortable or unable to specify a point ℓ_1 below which he/she <u>exclusively</u> prefers the minimization model, or a point r_1 above which he/she <u>exclusively</u> prefers the additive model. Second, the piecewise-linear function given by eq. (14) is not differentiable at the endpoints ℓ_1 and r_1. These two drawbacks argue for replacing $w_{\mathsf{MA1}}(m)$ with a "smooth" function that asymptotically approaches zero (resp. one) as the value of *m* decreases (resp. increases) while, at the same time, retains the desirable characteristics of $w_{\mathsf{MA1}}(m)$. A convenient function satisfying these requirements is a "logistic" function (i.e., a "sigmoid" function or "S-shaped" function). Hence, we refer to this extension as the *logistic min-additive model*. The logistic min-additive utility function, denoted as $\mathsf{MA}_2(\underline{\boldsymbol{u}}(\underline{x}))$, is specified as

$$\mathsf{MA}_2(\underline{\boldsymbol{u}}(\underline{x})) = \left(1 - w_{\mathsf{MA2}}(m)\right) \cdot \mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \; + \; w_{\mathsf{MA2}}(m) \cdot \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \tag{18}$$

where $\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ and $\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ are defined in eqs.(1) and (3), respectively, and $w_{\mathsf{MA2}}(m)$ is given by a logistic function[3] of the form (see Figure 6)

$$w_{\mathsf{MA}_2}(m) = \frac{1}{1 + \exp\left(-4 \cdot \left(\frac{m - \alpha_2}{\beta_2}\right)\right)} \tag{19}$$

[3] The more common form of logistics function is

$$\frac{1}{1 + \exp\left(\frac{m - \alpha}{\beta}\right)}$$

The coefficient –4 is used in eq. (19) in order to keep the notation used in the logistic min-additive model consistent with the notation used in the other versions of min-additive model.

where $\exp(\,)$ denotes exponentiation of the Euler number, e. The parameters α_2 and β_2 in the logistics MA weight function play analogous roles to the parameters α_1 and β_1 in the uniform MA weight function (see eq. (14)). Namely, α_2 is a "measure of location" and β_2 is a "measure of spread." For the logistic MA weight function, we define two additional parameters, denoted ℓ_2 and r_2, where ℓ_2 is the point below which DM "strongly prefers"[4] the minimization model $\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$; and r_2 is the point at which DM "strongly prefers"[5] the additive model $\mathtt{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$. Then α_2 is defined as the mid-point between ℓ_2 and r_2; and β_2 is defined as the difference between ℓ_2 and r_2. Thus, the following relationships hold:

$$\alpha_2 = \tfrac{1}{2}\cdot(r_2+\ell_2) \quad , \quad \beta_2 = r_2 - \ell_2 \quad , \quad \ell_2 = \alpha_2 - \tfrac{1}{2}\cdot\beta_2 \quad , \quad r_2 = \alpha_2 + \tfrac{1}{2}\cdot\beta_2 \tag{20}$$

These relationships enable us to describe the phrase "strongly prefers" (used in the definition of ℓ_2 and r_2) more precisely. Specifically, substituting, respectively, ℓ_2 and r_2 for m in eq. (19) yields

$$1 - w_{\mathsf{MA}_2}(\ell_2) = \frac{1}{1+e^{2}} \cong 0.88 \quad \text{and} \quad w_{\mathsf{MA}_2}(r_2) = \frac{1}{1+e^{-2}} \cong 0.88 \tag{21}$$

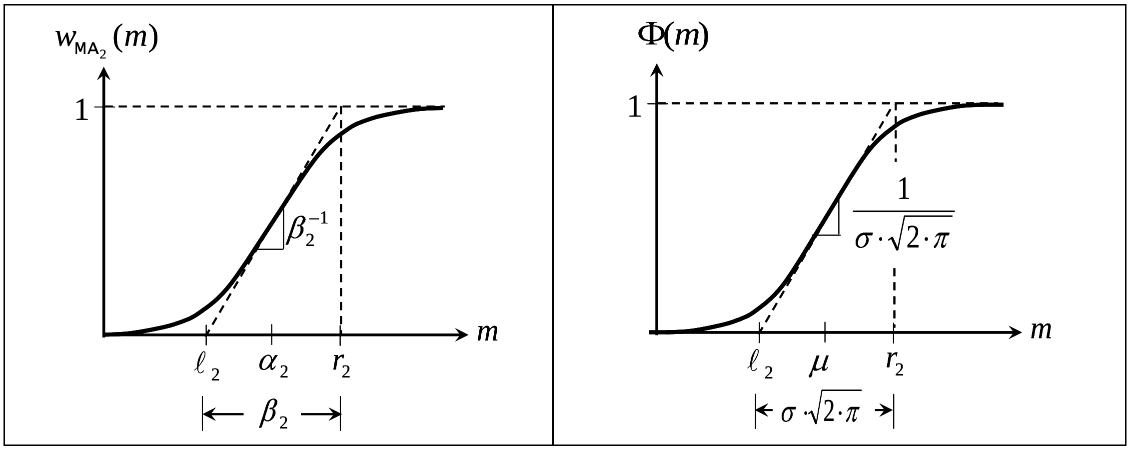
Thus, at $m = \ell_2$, there is an 88% chance that the DM will prefer the minimization model whereas, at $m = r_2$, there is an 88% chance that the DM will prefer the additive model.

Moreover, again analogous with the uniform min-additive model, β_2^{-1} (i.e., the reciprocal of β_2) is a "measure of steepness" of the distribution because β_2^{-1} is equal to the maximum rate of change of $w_{\mathsf{MA}2}(m)$ with respect to m. To show this "steepness" property, note that the derivative of $w_{\mathsf{MA}2}(m)$ with respect to m is given by

$$\partial w_{\mathsf{MA}_2}(m)/\partial m = \frac{2}{\beta_2\cdot\left(1+\cosh\left(-4\cdot\left(\dfrac{m-\alpha_2}{\beta_2}\right)\right)\right)} \tag{22}$$

where $\cosh(\,)$ denotes the hyperbolic cosine function. Observe that $\cosh(0)=1$ and that $1<\cosh(y)<\infty$ for all $y \neq 0$. Applying the hyperbolic cosine properties to eq. (22) shows that the maximum value of $\partial w_{\mathsf{MA}2}(m)/\partial m$ is attained at $m=\alpha_2$ and that $\partial w_{\mathsf{MA}2}(\alpha_2)/\partial m = \beta_2^{-1}$. Furthermore, these properties show that $\partial w_{\mathsf{MA}2}(m)/\partial m$ exists for all m in the domain $(-\infty,+\infty)$ provided that $\beta_2 \neq 0$. This condition on β_2, in turn, implies that the only requirement on ℓ_2 and r_2 is that $\ell_2 \neq r_2$. Thus, as with the uniform min-additive model, there is a great deal of flexibility in setting the parameters of the model.

[4] Defined below.
[5] Ibid.



(a) Logistic MA weight function (b) Gaussian CDF

Figure 7–Comparison of the logistic MA weight function and the Gaussian CDF

Finally, Cramer (2003) and others have noted the similarity between a logistic function and a Gaussian (i.e., Normal) cumulative distribution function (CDF) (see Figure 7). Let $\Phi(m)$ denote a Gaussian CDF and let $\phi(m) = \partial\Phi(m)/\partial m$ be a Gaussian density function given by

$$\phi(m) = \frac{1}{\sigma\cdot\sqrt{2\cdot\pi}}\cdot\exp\left(-\tfrac{1}{2}\left(\frac{m-\mu}{\sigma}\right)^2\right) \tag{23}$$

where μ is the mean, σ is the standard deviation, and $\exp(\,)$ denotes exponentiation of the Euler number, e. Note that μ is also the point of symmetry and that, at $m = \mu$, we have $\phi(\mu) = 1/(\sigma\cdot\sqrt{2\cdot\pi}) \cong 1/(2.5\cdot\sigma)$. Equating eqs. (22) and (23) at their point of symmetry yields

$$\mu = \tfrac{1}{2}\cdot(r_2 + \ell_2) = \alpha_2 \quad \text{and} \quad \sigma = \frac{r_2 - \ell_2}{\sqrt{2\cdot\pi}} = \frac{\beta_2}{\sqrt{2\cdot\pi}} \cong \frac{\beta_2}{2.5} = \tfrac{2}{5}\cdot\beta_2 \tag{24}$$

In other words, the logistic MA weight function can be approximated by the CDF of a Gaussian distribution with a mean of $\mu = \alpha_2$ and a standard deviation of approximately $\sigma \cong \frac{2}{5}\cdot\beta_2$.

The parameters of the min-additive model can also be set to represent a two-dimensional multi-linear model. This version of the min-additive model is discussed next.

5. Relaxed Min-Additive Model

As noted in Section 3, the additive and minimization models are special cases of the uniform min-additive model (see eq. (17)). In this section we show that a two dimensional (i.e., two attribute) multi-linear model is also a special case of the min-additive model. The general *n*-dimensional form of the multi-linear model was given in eq. (7). The two-dimensional version [Keeney and Raiffa (1976), pp. 233–235] of this equation is

$$\texttt{MULTILIN}(\underline{\boldsymbol{u}}(\underline{x})) = w_1 \cdot u_1(x_1) \; + \; w_2 \cdot u_2(x_2) \; + \; w_{12} \cdot u_1(x_1) \cdot u_2(x_2) \tag{25}$$

where w_1, w_2, and w_{12} are weights (i.e., constants). and $u_1(x_1)$ and $u_2(x_2)$ are single dimensional (i.e., single attribute) utility functions. In the multi-linear model, the first two weights, w_1 and w_2, are non-negative constants; and the three weights sum to one. That is, w_{12} = $1 - w_1 - w_2$. If w_{12} equals zero, then the multi-linear model reverts to a pure additive model and, as shown in Section 3, the uniform min-additive model is equivalent to the additive model if $r_1 \le 0$ and $\ell_1 < r_1$ (see eq. (17)).

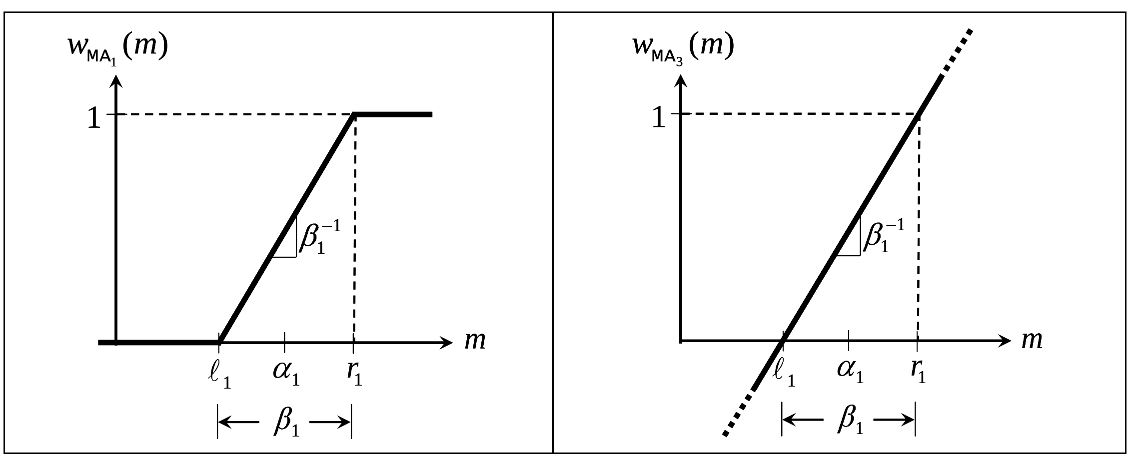
On the other hand, if $w_{12} \ne 0$, then this weight is permitted to take on negative as well as positive values. This relaxation is equivalent to permitting the uniform MA-weight function, $w_{\mathsf{MA}_1}(m)$, (see eq. (14)) to take on values outside the range $[0,1]$. We let $w_{\mathsf{MA3}}(m)$ denote the relaxed version of the MA weight function where $w_{\mathsf{MA3}}(m)$ is simply (see Figure 8)

$$w_{\mathsf{MA}_3}(m) = \tfrac{1}{2} + \frac{m - \alpha_1}{\beta_1} \tag{26}$$

This version of the model, called the *relaxed min-additive model*, is denoted by $\mathsf{MA}_3(\underline{\boldsymbol{u}}(\underline{x}))$ and is specified by

$$\mathsf{MA}_3(\underline{\boldsymbol{u}}(\underline{x})) = \left(1 - w_{\mathsf{MA3}}(m)\right) \cdot \texttt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \; + \; w_{\mathsf{MA3}}(m) \cdot \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \tag{27}$$

where $\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ and $\texttt{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ are defined in eqs.(1) and (3), respectively, and $w_{\mathsf{MA3}}(m)$ is given by eq. (26). Comparing eq. (14) in Section 3 with eq. (26) in this section, we see that the uniform MA weight function, $w_{\mathsf{MA1}}(m)$, is composed of three piecewise-linear segments whereas the relaxed MA weight function, $w_{\mathsf{MA3}}(m)$, is comprised of a single linear segment (again, see Figure 8).



(a) Uniform MA weight function (b) Relaxed MA weight function

Figure 8–Comparison of uniform and relaxed MA weight functions

Substituting the right-hand-side of eq. (26) for $w_{\mathsf{MA}3}(m)$ in eq. (27) and expanding the equation yields

$$\begin{aligned}\mathsf{MA}_3(\underline{\boldsymbol{u}}(\underline{x})) = \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x})) &+ \left(\tfrac{1}{2} - \frac{\alpha_1}{\beta_1}\right)\cdot\left(\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) - \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))\right) \\ &+ \left(\frac{1}{\beta_1}\right)\cdot\left(\mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))\right)\cdot\left(\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) - \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))\right)\end{aligned} \tag{28}$$

To cast eq. (28) in the form of the multi-linear model given in eq. (25), we consider four possible cases (depending on whether w_1 and/or w_2 equals zero). Note that the case where w_{12} equals zero has already been covered. Thus, in Case 1 through Case 4 below, we assume that $w_{12} \neq 0$.

Case 1: $w_1 \neq 0$, $w_2 \neq 0$ and $w_{12} \neq 0$

If all three weights are non-zero, then we make the following assignments:

$$\mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x})) = w_1 \cdot u_1(x_1) \quad \text{and} \quad \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) = w_1 \cdot u_1(x_1) + w_2 \cdot u_2(x_2) \tag{29a}$$

$$\beta_1 = \frac{w_1 \cdot w_2}{w_{12}} \quad \text{and} \quad \alpha_1 = -\tfrac{1}{2}\beta_1 \tag{29b}$$

Substituting the expressions in eq. (29) into the min-additive model specified in eq. (28) gives

$$\begin{aligned}\mathsf{MA}_3(\underline{\boldsymbol{u}}(\underline{x})) &= \left(w_1 \cdot u_1(x_1)\right) + \left(\tfrac{1}{2} - \frac{-\tfrac{1}{2}\cdot\beta_1}{\beta_1}\right)\cdot\left(w_1 \cdot u_1(x_1) + w_2 \cdot u_2(x_2) - w_1 \cdot u_1(x_1)\right) \\ &\quad + \left(\frac{w_{12}}{w_1 \cdot w_2}\right)\cdot\left(w_1 \cdot u_1(x_1)\right)\cdot\left(w_1 \cdot u_1(x_1) + w_2 \cdot u_2(x_2) - w_1 \cdot u_1(x_1)\right) \\ &= \left(w_1 \cdot u_1(x_1)\right) + \left(w_2 \cdot u_2(x_2)\right) + w_{12}\cdot\left(u_1(x_1)\right)\cdot\left(u_2(x_2)\right) \\ &= \mathsf{MULTILIN}(\underline{\boldsymbol{u}}(\underline{x}))\end{aligned} \tag{30}$$

Case 2: $w_1 = 0$, $w_2 \neq 0$ and $w_{12} \neq 0$

If w_1 equals zero but the other weights are non-zero, then we make the following assignments:

$$\mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x})) = w_2 \cdot u_2(x_2) \quad \text{and} \quad \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) = u_1(x_1) + w_2 \cdot u_2(x_2) \tag{31a}$$

$$\beta_1 = \frac{w_2}{w_{12}} \quad \text{and} \quad \alpha_1 = \tfrac{1}{2}\beta_1 \tag{31b}$$

Substituting the expressions in eq. (31) into the min-additive model specified in eq. (28) gives

$$\begin{aligned}\mathtt{MA}_3(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})) &= \left(w_2 \cdot u_2(x_2)\right) + \left(\tfrac{1}{2} - \frac{\frac{1}{2}\cdot\beta_1}{\beta_1}\right)\cdot\left(u_1(x_1) + w_2\cdot u_2(x_2) - w_2\cdot u_2(x_2)\right)\\ &\quad + \left(\frac{w_{12}}{w_2}\right)\cdot\left(w_2\cdot u_2(x_2)\right)\cdot\left(u_1(x_1) + w_2\cdot u_2(x_2) - w_2\cdot u_2(x_2)\right)\\ &= \left(w_2\cdot u_2(x_2)\right) + w_{12}\cdot\left(u_1(x_1)\right)\cdot\left(u_2(x_2)\right)\\ &= \mathtt{MULTILIN}(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}}))\end{aligned} \tag{32}$$

Case 3: $w_1 \neq 0,\ w_2 = 0$ and $w_{12} \neq 0$

If w_2 equals zero but the other weights are non-zero, then we make the following assignments:

$$\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})) = w_1\cdot u_1(x_1) \quad \text{and} \quad \mathtt{ADD}(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})) = w_1\cdot u_1(x_1) + u_2(x_2) \tag{33a}$$

$$\beta_1 = \frac{w_1}{w_{12}} \quad \text{and} \quad \alpha_1 = \tfrac{1}{2}\beta_1 \tag{33b}$$

Substituting the expressions in eq. (33) into the min-additive model specified in eq. (28) gives

$$\begin{aligned}\mathtt{MA}_3(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})) &= \left(w_1 \cdot u_1(x_1)\right) + \left(\tfrac{1}{2} - \frac{\frac{1}{2}\cdot\beta_1}{\beta_1}\right)\cdot\left(u_2(x_2) + w_1\cdot u_1(x_1) - w_1\cdot u_1(x_1)\right)\\ &\quad + \left(\frac{w_{12}}{w_1}\right)\cdot\left(w_1\cdot u_1(x_1)\right)\cdot\left(w_1\cdot u_1(x_1) + u_2(x_2) - w_1\cdot u_1(x_1)\right)\\ &= \left(w_1\cdot u_1(x_1)\right) + w_{12}\cdot\left(u_1(x_1)\right)\cdot\left(u_2(x_2)\right)\\ &= \mathtt{MULTILIN}(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}}))\end{aligned} \tag{34}$$

Case 4: $w_1 = 0,\ w_2 = 0$ and $w_{12} \neq 0$

Finally, if both w_1 and w_2 equal zero, then we make the following assignments:

$$\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})) = u_1(x_1) \quad \text{and} \quad \mathtt{ADD}(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})) = 1 + u_1(x_1) - u_2(x_2) \tag{35a}$$

$$\beta_1 = -1 \quad \text{and} \quad \alpha_1 = \tfrac{1}{2}\beta_1 \tag{35b}$$

Substituting the expressions in eq. (35) into the min-additive model specified in eq. (28) gives

$$\begin{aligned}\mathtt{MA}_3(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})) &= \left(u_1(x_1)\right) + \left(\tfrac{1}{2} - \frac{\frac{1}{2}\cdot\beta_1}{\beta_1}\right)\cdot\left(1 + u_1(x_1) - u_2(x_2) - u_1(x_1)\right)\\ &\quad - \left(u_1(x_1)\right)\cdot\left(1 - u_2(x_2)\right)\\ &= \left(u_1(x_1)\right) - \left(u_1(x_1)\right)\cdot\left(1 - u_2(x_2)\right)\\ &= \left(u_1(x_1)\right)\cdot\left(u_2(x_2)\right)\\ &= \mathtt{MULTILIN}(\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}}))\end{aligned} \tag{36}$$

By considering these four cases, we have shown that *any* two dimensional multi-linear model can be converted into the relaxed min-additive model. Hence, this version of the min-additive model inherits all the properties associated with the multi-linear model (e.g., connectivity, transitivity, mutual utility independence, etc. [Keeney and Raiffa (1976), pp. 232–240]). Moreover, note that in Case 4 above, the relaxed min-additive model was equated to a two-dimensional multiplicative model. Thus, this version of the min-additive model is a generalization of both the multi-linear utility function and the multiplicative utility function.

6. Graphical Representation and Numerical Example

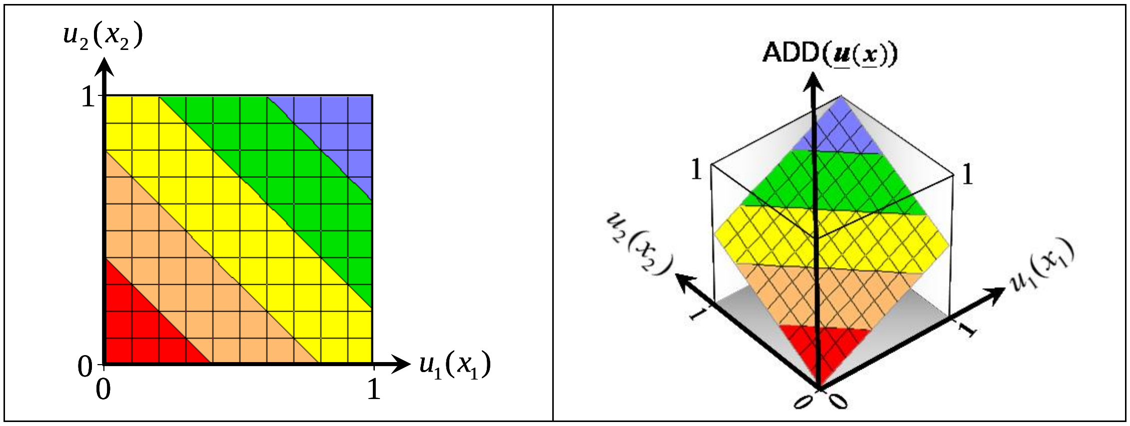
This section provides a visual representation of the models presented in this paper. The material is divided into three parts. Section 6.1 graphically compares the basic MA model with the additive and minimization models. Section 6.2 illustrates the effects of the "location" and "spread" parameters using the uniform and logistic MA models. Section 6.3 visually demonstrates the equivalence between the two dimensional multi-linear model and the relaxed MA model. [Additional graphical representations are contained in Appendix B.]

6.1 Additive, Minimization, and Basic Min-Additive Models

To visualize the basic form of the min-additive model, consider again the example of a project manager (the DM) who is considering two attributes—project cost (x_1) and project schedule (x_2). The cost attribute can vary from x_1^{BEST}(project completed in budget) to x_1^{WORST}(project suffers unacceptable cost overruns); and the schedule attribute can vary from x_2^{BEST}(project completed on schedule) to x_2^{WORST}(project suffers unacceptable delays). We set $u_1(x_1^{\mathsf{WORST}})=0$, $u_1(x_1^{\mathsf{BEST}})=1$, $u_2(x_2^{\mathsf{WORST}})=0$, and $u_2(x_2^{\mathsf{BEST}})=1$. For this example, we assume that the two single dimensional utility functions are weighted equally (i.e., $w_1=\frac{1}{2}$ and $w_2=\frac{1}{2}$). Figures 9, 10, and 11 show topographic projections of, respectively, the additive model (eq. (1)), the minimization model (eq. (3)), and the basic min-additive model (eq. (9)). Each figure contains (a) an overhead view (i.e., plan view) showing contours projected onto the $u_1 \times u_2$ plane; and (b) a perspective view showing an isometric sketch contained within the unit-cube. The color bands in the figures represent intervals of utility values. The contour lines on the boundary between adjacent color bands represent iso-utility curves (i.e., indifference curves). The legend for the color bands is summarized in Table 1.

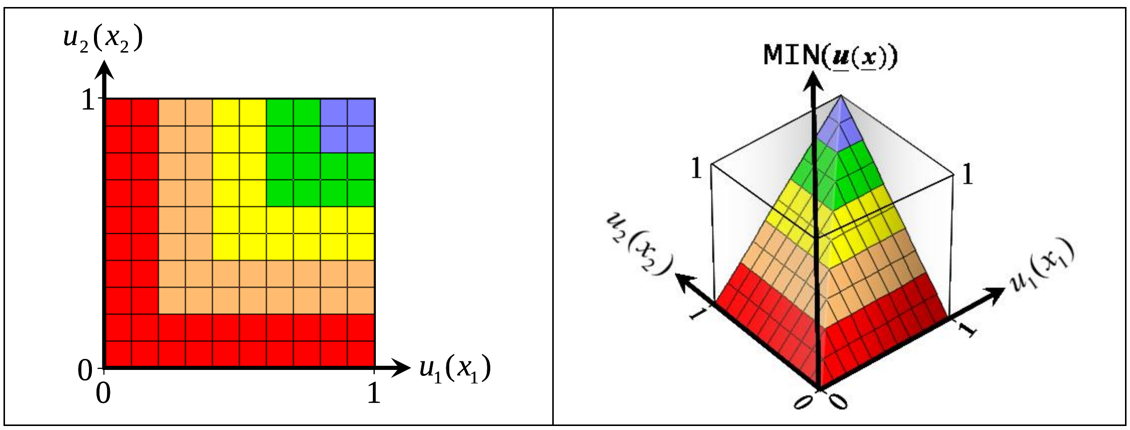
Color	Swatch	Utility value interval
Blue		[0.8 , 1.0]
Green		[0.6 , 0.8]
Yellow		[0.4 , 0.6]
Orange		[0.2 , 0.4]
Red		[0.0 , 0.2]

Table 1–Legend for topographic contours



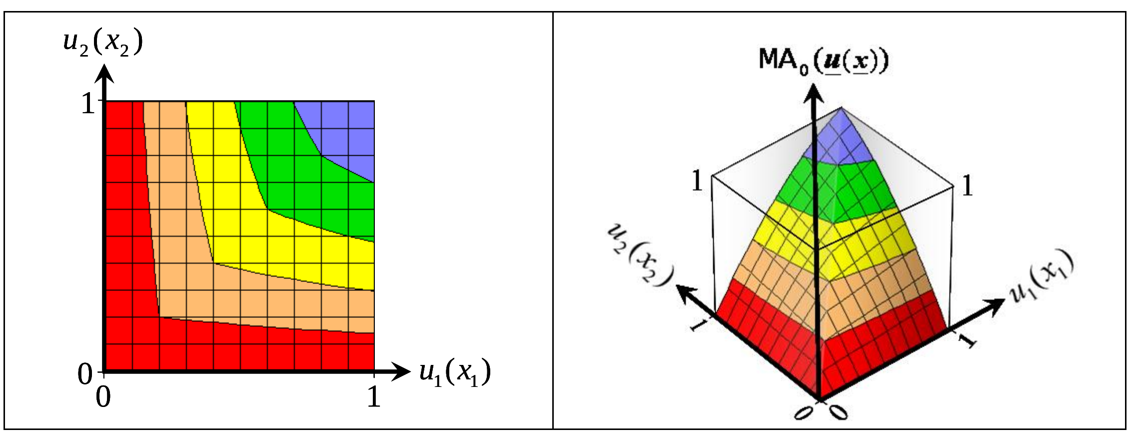
(a) Overhead view (b) Perspective view

Figure 9–Graphic representation of additive model



(a) Overhead view (b) Perspective view

Figure 10–Graphic representation of minimization model



(a) Overhead view (b) Perspective view

Figure 11–Graphic representation of basic min-additive model

Comparing Figure 11 with Figures 9 and 10, we see that the basic MA resembles the additive model if both individual utility values are in the proximity of one. On the other hand, the shape of the basic MA model is similar to the minimization model if either one (or both) of the individual utility values is in the proximity of zero.

The topographic projections can also be used to visualize the four alternatives discussed in Sections 1 and 2 (see Figures 1 and 2). Figure 12 plots the coordinates of alternatives A, B, C, and D on the $u_1 \times u_2$ plane.

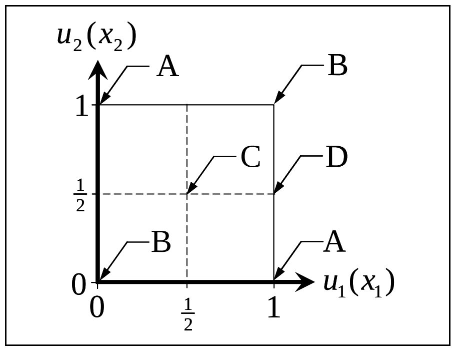
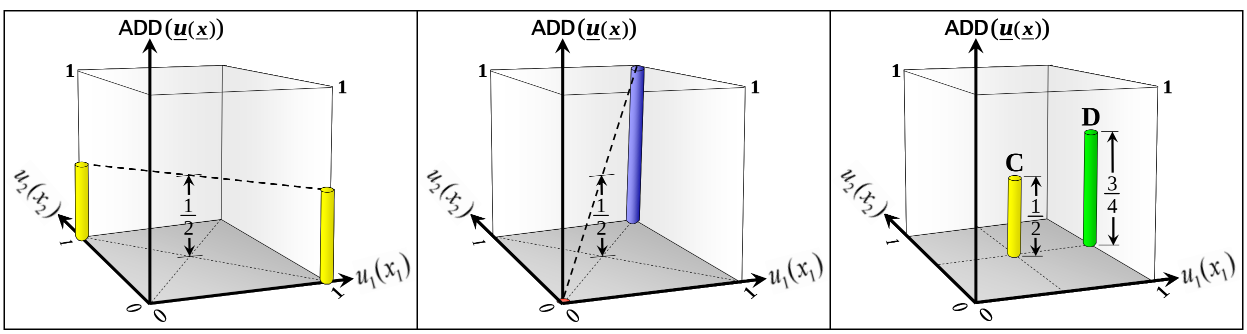
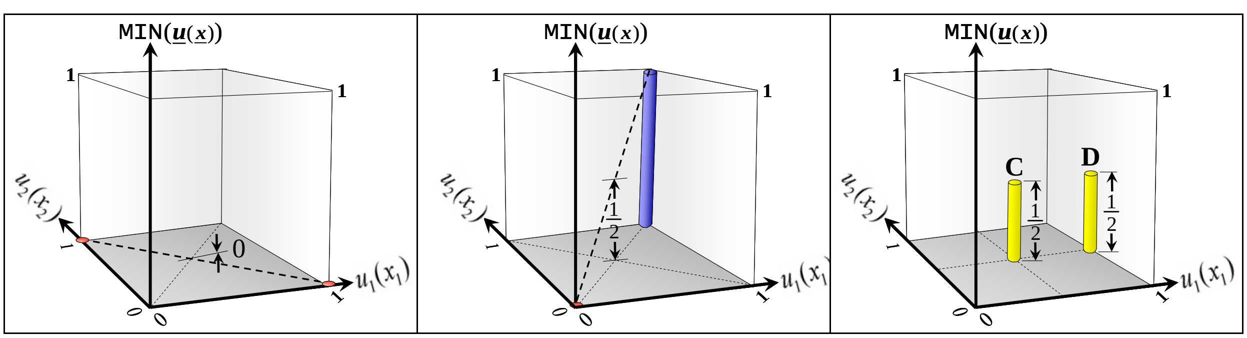


Figure 12–Coordinates of alternatives A, B, C, and D



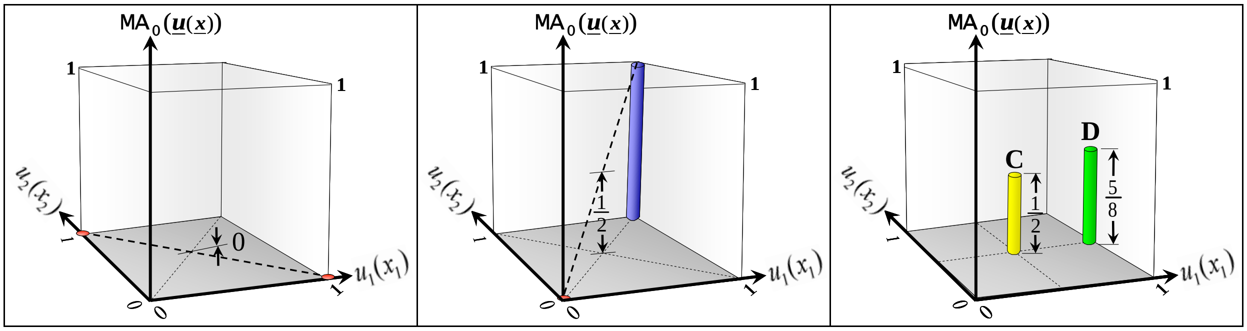
(a) Alternative A (b) Alternative B (c) Alternatives C and D

Figure 13–Value of alternatives A, B, C, an D using additive model



(a) Alternative A (b) Alternative B (c) Alternatives A and D

Figure 14–Value of alternatives A, B, C, an D using minimization model



(a) Alternative A (b) Alternative B (c) Alternatives C and D

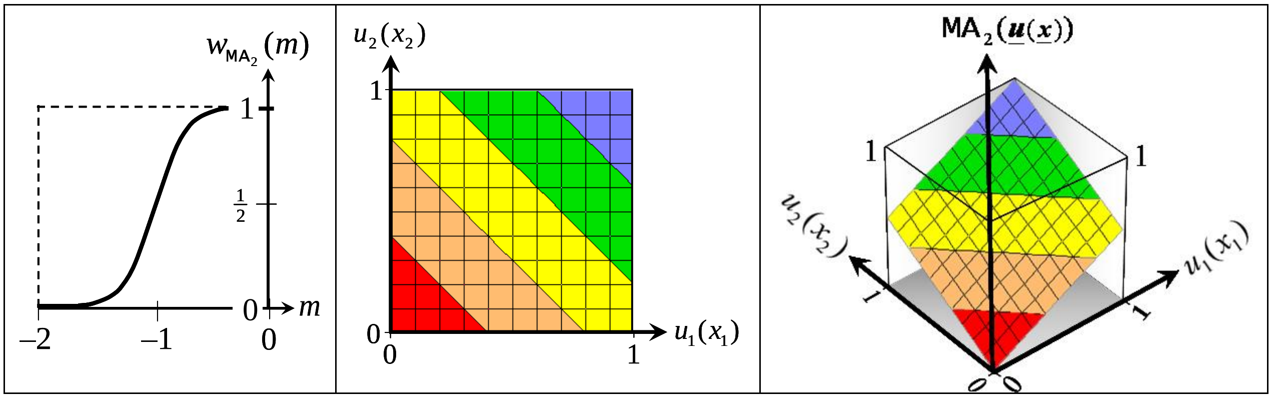
Figure 15–Value of alternatives A, B, C, an D using basic min-additive model

Figure 13 displays the topographic projection of these four alternatives based on the additive model. This figure shows that the additive model gives the same utility value for alternative A and B. Figure 14 displays the topographic projection of the four alternatives based on the minimization model. This figure shows that the minimization model does not distinguish between alternatives C and D. Figure 15 displays the topographic projection of the four alternatives using the basic MA model. This figure shows that the basic MA model represents the DM's preference for alternative B over alternative A as well as the DM's preference for alternative D over alternative C.

6.2 Logistic Mid-Additive and Uniform Min-Additive Models

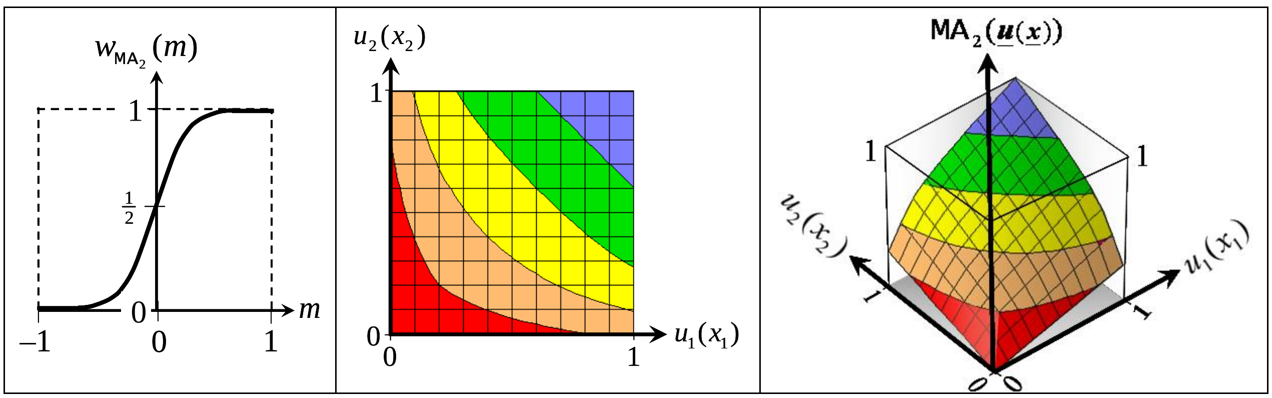
This section illustrates the effects of the "location" and "steepness" parameters using the same project management example that was summarized in Section 6.1. The topographic views shown in this section are based on the logistic MA model. Very similar contours are produced by the uniform MA model. Figures 16 through 18 illustrate the effects of alternating the "location" parameter, α_2, ceteris paribus. Specifically, the "steepness" parameter, β_2^{-1} is held fixed at $\frac{1}{2}$ while α_2 is varied from –1 to +1. At $\alpha_2 = -1$ (see Figure 16), the MA model is almost identical to the additive model (see Figure 9), at $\alpha_2 = +1$ (see Figure 18), the MA model is very similar to the minimization model (see Figure 10); and at $\alpha_2 = 0$ (see Figure 17), the MA model is a mixture of the two extremes represented by the additive and minimization models.

Figures 19 through 21 show the effects of alternating the "steepness" parameter, β_2^{-1}, ceteris paribus. Specifically, the "location" parameter, α_2 is held fixed at $\frac{1}{2}$ while β_2^{-1} is varied from 0.2 to 20. At $\beta_2^{-1} = 0.2$ (see Figure 19), the MA weight function has a very gentle gradient. A gentle gradient means that the weight placed on the additive model is just slightly less than $\frac{1}{2}$ for low values of *m*; and just slightly more than $\frac{1}{2}$ for high values of *m*. In contrast, at $\beta_2^{-1} = 20$ (see Figure 21), the MA weight function has a very steep gradient near $m = \frac{1}{2}$. A steep gradient means that the weight placed on the additive model changes abruptly for values of *m* near $\frac{1}{2}$ with almost no weight placed on the additive model for $m < \frac{1}{2}$ and almost 100% weight placed on the additive model for $m > \frac{1}{2}$. At $\beta_2^{-1} = 2.0$, (see Figure 20) the logistic (and uniform) MA model performs in a manner similar to the basic MA model (compare Figures 11 and 20).



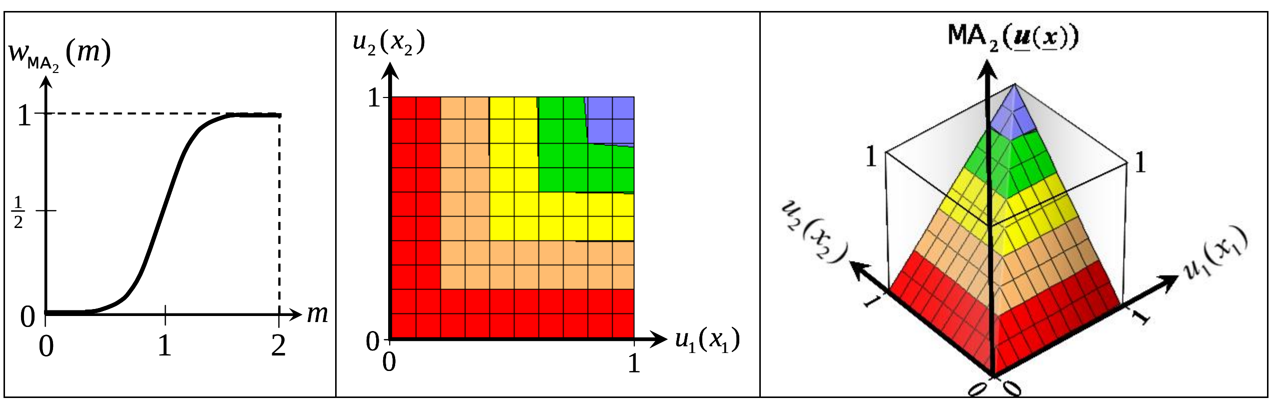
(a) MA weight function (b) Overhead view (c) Perspective view

Figure 16–Logistic MA model with $\alpha_2 = -1$ and $\beta_2^{-1} = 2$ (i.e., $\beta_2 = \frac{1}{2}$)



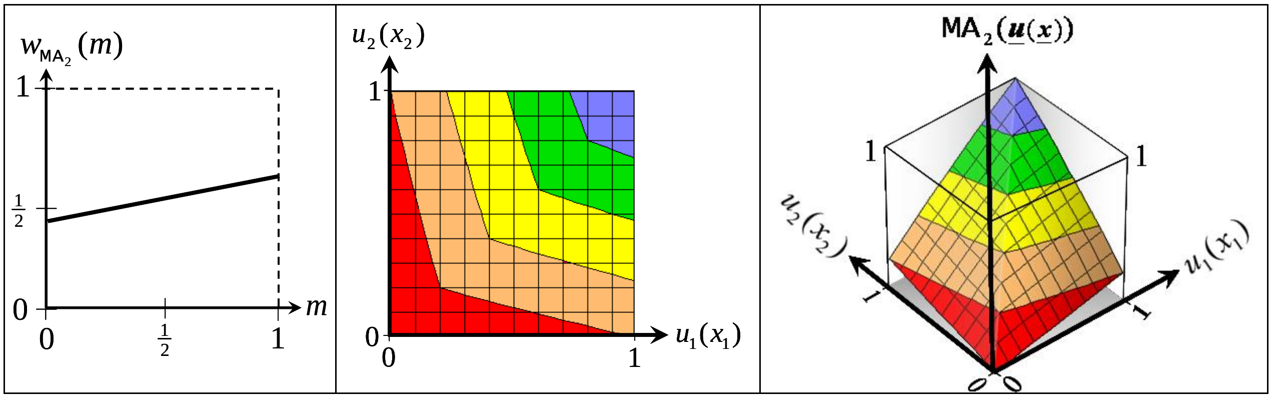
(a) MA weight function (b) Overhead view (c) Perspective view

Figure 17–Logistic MA model with $\alpha_2 = 0$ and $\beta_2^{-1} = 2$ (i.e., $\beta_2 = \frac{1}{2}$)



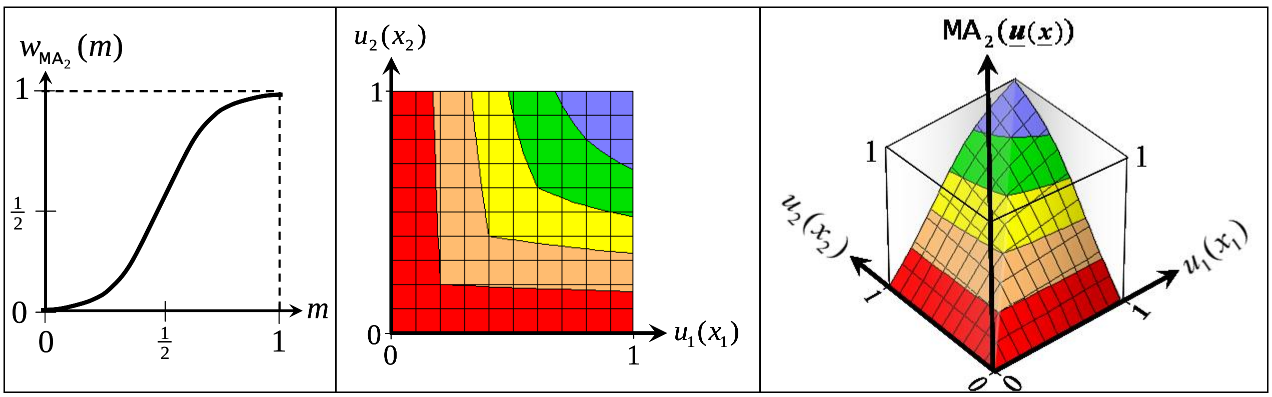
(a) MA weight function (b) Overhead view (c) Perspective view

Figure 18–Logistic MA model with $\alpha_2 = +1$ and $\beta_2^{-1} = 2$ (i.e., $\beta_2 = \frac{1}{2}$)



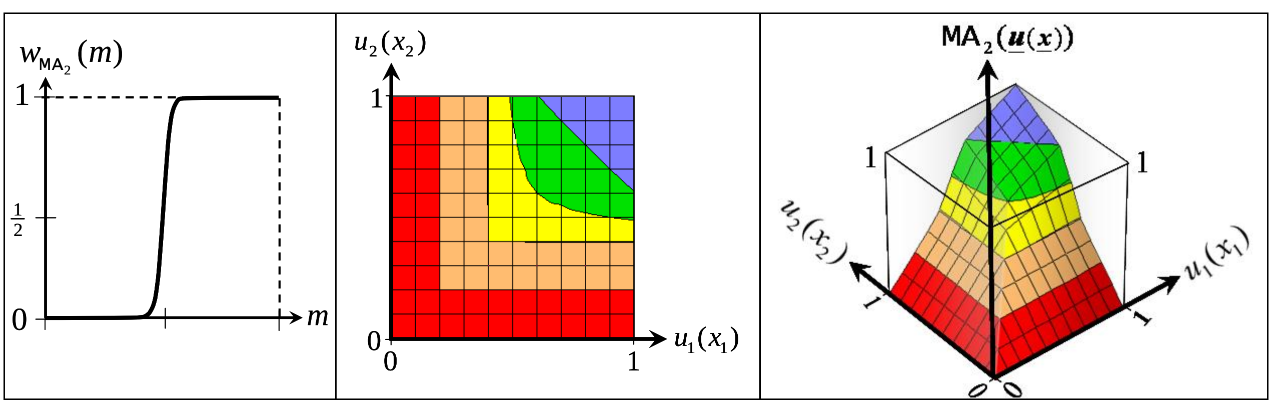
(a) MA weight function (b) Overhead view (c) Perspective view

Figure 19–Logistic MA model with $\alpha_2 = \frac{1}{2}$ and $\beta_2^{-1} = 0.2$ (i.e., $\beta_2 = 5$)



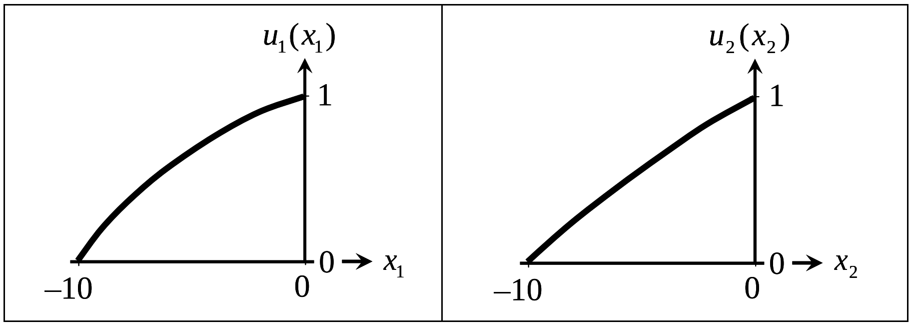
(a) MA weight function (b) Overhead view (c) Perspective view

Figure 20–Logistic MA model with $\alpha_2 = \frac{1}{2}$ and $\beta_2^{-1} = 2$ (i.e., $\beta_2 = \frac{1}{2}$)



(a) MA weight function (b) Overhead view (c) Perspective view

Figure 21–Logistic MA model with $\alpha_2 = \frac{1}{2}$ and $\beta_2^{-1} = 20$ (i.e., $\beta_2 = \frac{1}{20}$)



(a) Shortage attribute (b) Outdating attribute

Figure 22–Single attribute utility functions for blood bank example

6.3 Multi-Linear Model and Relaxed Min-Additive Model

This section illustrates the equivalence between the two dimensional multi-linear model and the relaxed min-additive model using an example involving policy decisions for inventorying whole blood, blood plasma, and other components at a hospital blood bank [Jennings (1968) pp. 335–342], [Keeney and Raiffa (1976), pp. 273–281], [Clemen and Reilly (2004), pp. 587–581]. Since demand for blood units is stochastic, a safety stock must be maintained to minimize the probability of shortages. Yet, blood units are also perishable and must be discarded if their allowable shelf-life is exceeded.[6] Let x_1.be the annual percent of units demanded but not in stock; and let x_2 be the annual percent of units removed from inventory due to outdating. The nurse in charge of maintaining blood supplies at the hospital (the DM) has determined that $x_1^{\text{WORST}} = -10$ (i.e., a 10% shortage rate) and $x_1^{\text{BEST}} = 0$ (i.e., no shortage of blood units). Also, $x_2^{\text{WORST}} = -10$ (i.e., a 10% discard rate) and $x_2^{\text{BEST}} = 0$ (i.e., no units discarded due to outdating). Her single attribute utility functions were assessed as

$$u_1(x_1) = \frac{e^{-0.13 \cdot x_1} - e^{-0.13 \cdot x_1^{\text{WORST}}}}{e^{-0.13 \cdot x_1^{\text{BEST}}} - e^{-0.13 \cdot x_1^{\text{WORST}}}} = \frac{3.6693 - e^{-0.13 \cdot x_1}}{2.6693} \qquad (37a)$$

$$u_2(x_2) = \frac{e^{-0.04 \cdot x_2} - e^{-0.04 \cdot x_2^{\text{WORST}}}}{e^{-0.04 \cdot x_2^{\text{BEST}}} - e^{-0.04 \cdot x_2^{\text{WORST}}}} = \frac{1.4918 - e^{-0.04 \cdot x_2}}{0.4918} \qquad (37b)$$

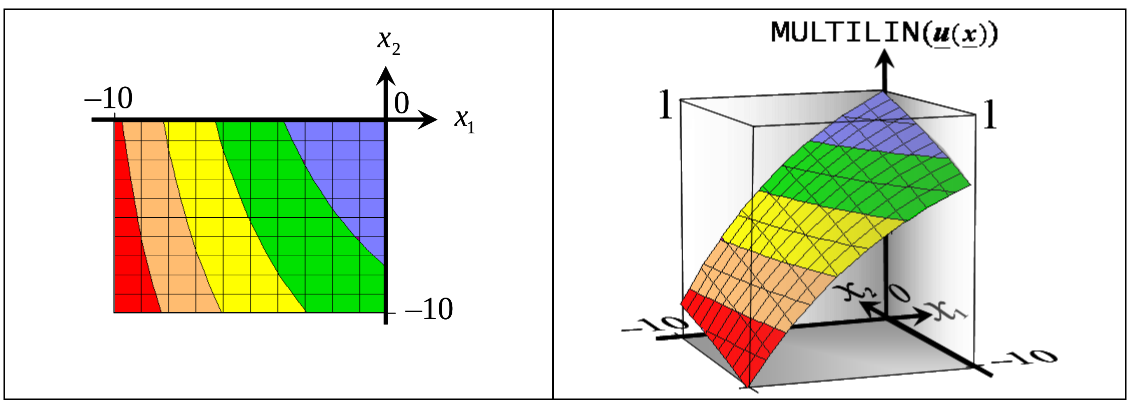
where e is the Euler number (i.e., $e \cong 2.718282$). The coefficients –0.13 and –0.04 indicate a greater disutility for shortages than for outdating (see Figure 22).

Assuming mutual utility independence [Keeney and Raiffa (1976), pp. 264–267], a two-dimensional multi-linear model (see eq. (25) was calibrated with $w_1 = 0.7242$, $w_1 = 0.1381$, and $w_{12} = 0.1377$. Using these weights and substituting $u_1(x_1)$ and $u_2(x_2)$ from eq. (37) into eq. (25) yields

[6] Maximum allowable storage time for most blood components is approximately three weeks.

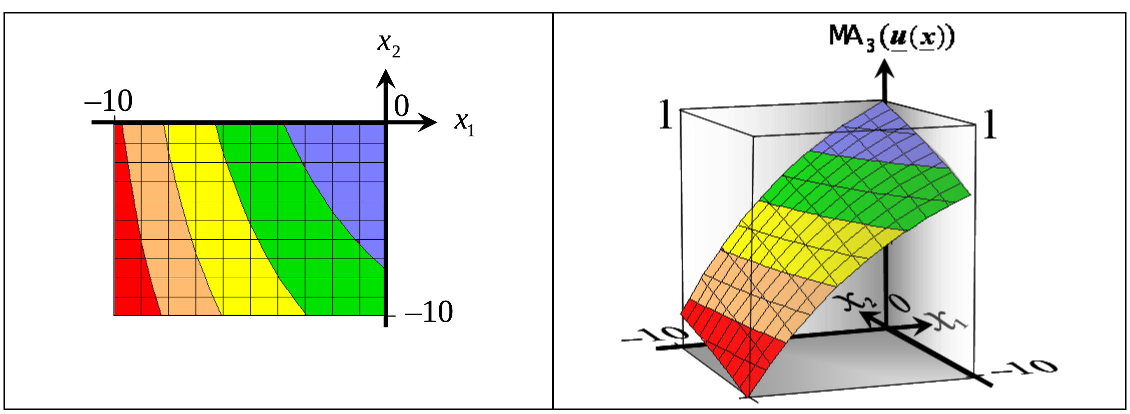
$$\texttt{MULTILIN}(\underline{\boldsymbol{u}}(\underline{x})) = 0.7242 \cdot \left(\frac{3.6693 - e^{-0.13 \cdot x_1}}{2.6693} \right) + 0.1381 \cdot \left(\frac{1.4918 - e^{-0.04 \cdot x_2}}{0.4918} \right) + 0.1377 \cdot \left(\frac{3.6693 - e^{-0.13 \cdot x_1}}{2.6693} \right) \cdot \left(\frac{1.4918 - e^{-0.04 \cdot x_2}}{0.4918} \right) \quad (38)$$

A topographic projection of eq. (38) is produced in Figure 23. Figure 23(a) shows the contour lines for the utility function projected onto the $x_1 \times x_2$ attribute plane; and Figure 23(b) is a 3-dimensional rendering of this function. Observe that the utility function decreases as either x_1 or x_2 decreases; and that the rate of decrease is greater for x_1 than for x_2 reflecting the DM's greater concern for shortages of blood rather than for the outdating of blood.



(a) Overhead view (b) Perspective view

Figure 23–Multi-linear model for blood bank example



(a) Overhead view (b) Perspective view

Figure 24–Relaxed min-additive model for blood bank example

To construct the equivalent representation using the relaxed min-additive model, note that all the weights in eq. (38) are non-zero. Hence, Case 1 in Section 5 applies and we set

$$\begin{aligned}
\texttt{MIN}(\underline{\boldsymbol{u}}(\underline{x})) &= w_1 \cdot u_1(x_1) = 0.7242 \cdot \left(\frac{3.6693 - e^{-0.13 \cdot x_1}}{2.6693} \right) \\
\texttt{ADD}(\underline{\boldsymbol{u}}(\underline{x})) &= w_1 \cdot u_1(x_1) + w_2 \cdot u_2(x_2) \\
&= 0.7242 \cdot \left(\frac{3.6693 - e^{-0.13 \cdot x_1}}{2.6693} \right) + 0.1381 \cdot \left(\frac{1.4918 - e^{-0.04 \cdot x_2}}{0.4918} \right) \\
\beta_1 &= \frac{w_1 \cdot w_2}{w_{12}} = \frac{(0.7242) \cdot (0.1381)}{0.1377} = 0.7263 \\
\alpha_1 &= -\tfrac{1}{2} \cdot \beta_1 = -\tfrac{1}{2} \cdot (0.7263) = -0.3632
\end{aligned} \tag{39}$$

By substituting the expressions in eq. (39) into the relaxed min-additive model specified in eq. (28) we get

$$\begin{aligned}
\texttt{MA}_3(\underline{\boldsymbol{u}}(\underline{x})) &= 0.7242 \cdot \left(\frac{3.6693 - e^{-0.13 \cdot x_1}}{2.6693} \right) + (\tfrac{1}{2} + \tfrac{1}{2}) \cdot 0.1381 \cdot \left(\frac{1.4918 - e^{-0.04 \cdot x_2}}{0.4918} \right) \\
&\quad + \left(\frac{1}{0.7263} \right) \cdot \left(0.7242 \cdot \left(\frac{3.6693 - e^{-0.13 \cdot x_1}}{2.6693} \right) \right) \cdot \left(0.1381 \cdot \left(\frac{1.4918 - e^{-0.04 \cdot x_2}}{0.4918} \right) \right) \\
&= 0.7242 \cdot \left(\frac{3.6693 - e^{-0.13 \cdot x_1}}{2.6693} \right) + 0.1381 \cdot \left(\frac{1.4918 - e^{-0.04 \cdot x_2}}{0.4918} \right) \\
&\quad + 0.1377 \cdot \left(\frac{3.6693 - e^{-0.13 \cdot x_1}}{2.6693} \right) \cdot \left(\frac{1.4918 - e^{-0.04 \cdot x_2}}{0.4918} \right)
\end{aligned} \tag{40}$$

Comparing eq. (38) with eq. (40), we see that the two equations are identical. Hence, $\texttt{MA}_3(\underline{\boldsymbol{u}}(\underline{x})) = \texttt{MULTILIN}(\underline{\boldsymbol{u}}(\underline{x}))$. This equivalence is also confirmed by comparing the topographic projection of eq. (38) (see Figure 23) with the topographic projection of eq. (40) (see Figure 24).

The next section summarizes the paper.

7. Summary

This paper has presented a family of "min-additive" (MA) utility functions that generalize the additive and minimization utility functions. The MA models can be employed in situations where the decision-maker's preferences violate the additive independence assumptions inherent in the additive utility model. The basic version of the MA model does not require any additional parameters to be estimated. Extensions of the basic model use "location" and "spread" parameters to specify a wide range of decision-makers' preferences. Moreover, these parameters can also be set to represent the two-dimensional multi-linear utility function (and the two-dimensional multiplicative utility function). Thus, the MA utility model is a generalization of a host of other utility models.

Extensions of the MA model presented in this paper are also possible. Appendix A shows how the MA utility function can be nested in a decision tree hierarchy; Appendix B compares the MA model with two other models—the "limited average" utility function and the "exponential-average" utility function; and Appendix C describes the "max-additive" function used to model risk minimization (rather than utility maximization). For further examples of analytical methods for risk management, see, for example, [Garvey (2000)], [Garvey (2009)].

In sum, these extensions, together with the four versions of the min-additive model presented in this paper can help decision-makers make better decisions.

Appendix A

This appendix illustrates how the min-additive model can be nested in a decision-making hierarchy. Consider the following hypothetical situation.[7] Mr. Taylor's classroom has three students in it: Xaviar (X), Yvonne (Y), and Zachary (Z). Each student is taking four subjects: ENGLISH, HISTORY, MATH, and SCIENCE. Suppose that the students have received the grades for the four subjects that are posted in Table 2.

Student	Subject and grade				Method of Summarizing		
	ENGLISH	HISTORY	MATH	SCIENCE	Grade point average	Lowest grade	Basic "Min-Additive" model
X	A	A	A	B	A–	B	A–
Y	B	B	B	B	B	B	B
Z	A	A	A	F	B	F	F

Table 2–Hypothetical subject grades and summary measures for three students

How should a student's grades be aggregated? The traditional method, of course, is to compute a grade point average (assigning 4 grade points for an "A," 3 grade points for a "B," etc., summing the grade points, and dividing by the number of subjects). The grade point average summary is shown in Table 2. The grade point average summary distinguishes between students X and Y but does not distinguish between students Y and Z. In particular, the grade point average summary does not reflect student Z's failing grade in SCIENCE. One method of capturing this failing grade in a summary measure is to report the lowest (i.e., minimum) grade for each student. Table 2 also

[7] This example is intended to illustrate the work-breakdown-structure (WBS) typical of many system-of-systems configurations. It is *not* intended to advocate any polices regarding educational assessment.

shows the lowest grade summary measure. This measure distinguishes between student Y and student Z but it does not distinguish between student X and student Y.

The basic "min-additive" (MA) summary measure (described in Section 2) distinguishes among all three students. Specifically, the grade point average summary measure is computed using $\mathtt{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ (see eq. (1)), the lowest grade summary measure is specified using $\mathtt{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ (see eq. (3)); and the basic "min-additive" (MA) summary measure is calculated using $\mathtt{MA}_0(\underline{\boldsymbol{u}}(\underline{x}))$ (see eq. (9)). As shown in Table 2, for students that are performing well (getting "A"'s and "B"'s), the MA summary measure places emphasis on the grade point average. However, if a student is receiving a failing grade in any subject, that information is not overlooked.

This same method of averaging high scores while "red flagging" low scores can be implemented at multiple levels as illustrated by the hypothetical hierarchy depicted in Figure 25. The Regional School (node "R") has two classrooms: Mr. Taylor's classroom (node "T") and Ms. Smith's classroom (node "S"). As mentioned before, Mr. Taylor's classroom has three students in it: Xaviar (X), Yvonne (Y), and Zachary (Z) (see Table 2). Ms. Smith's classroom also has three students in it: Ulysses (U), Victor (V), and Wilbert (W); and each student in Ms. Smith's class is also taking four subjects: ENGLISH, HISTORY, MATH, and SCIENCE. Starting at the bottom level of the tree, the basic MA summary measure is applied to each successively higher level. Thus, the students' "scores" are based on their course grades, teachers' "scores" are based on the scores of the students in their class, the school's "score" is based on the scores of the teachers in the school, and so on. By using the MA summary measure, overall performance is measured at each level without attenuating a failing score at a lower level. The measure also provides a "trace" to the "root cause" of a failing score (a rightmost depth first search for the tree hierarchy shown in Figure 25. Moreover, the MA performance measure can pinpoint where additional resources or other corrective measures can gainfully be employed.

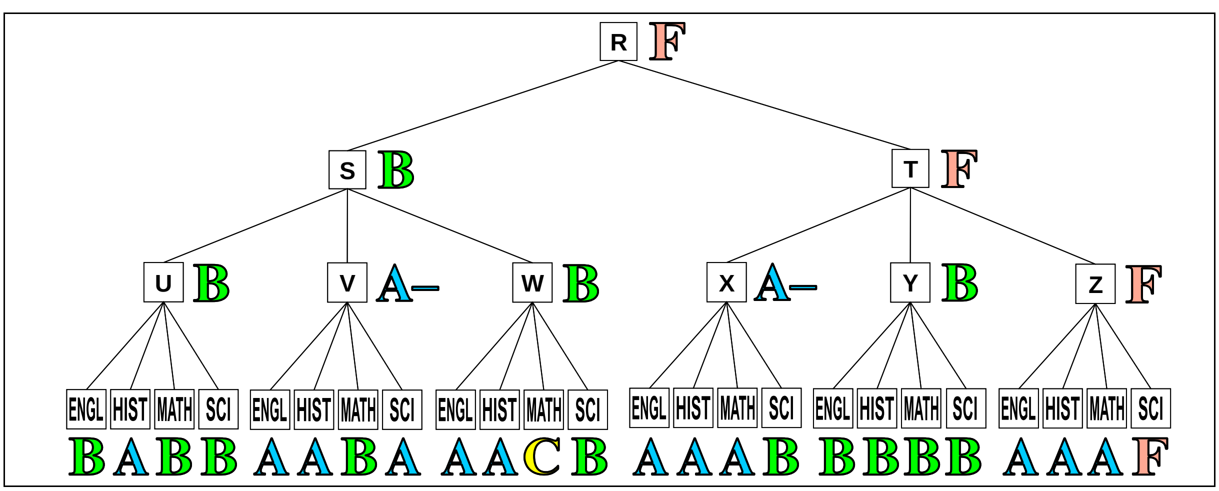


Figure 25–Hypothetical example of hierarchy of min-additive performance measures

Appendix B

This appendix summarizes other methods that combine the additive and minimization models. Two promising models of this type are the *limited average model* [Moynihan and Shimi (2004)] and the *exponential average model* [Schmidt (2007)].

B.1 Limited Average Model

The limited average model, denoted $\mathsf{LIMAVG}(\underline{\boldsymbol{u}}(\underline{x}))$, is given by

$$\mathsf{LIMAVG}(\underline{\boldsymbol{u}}(\underline{x})) = \min\{\, \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \;,\; \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x})) + \lambda \,\} \tag{41}$$

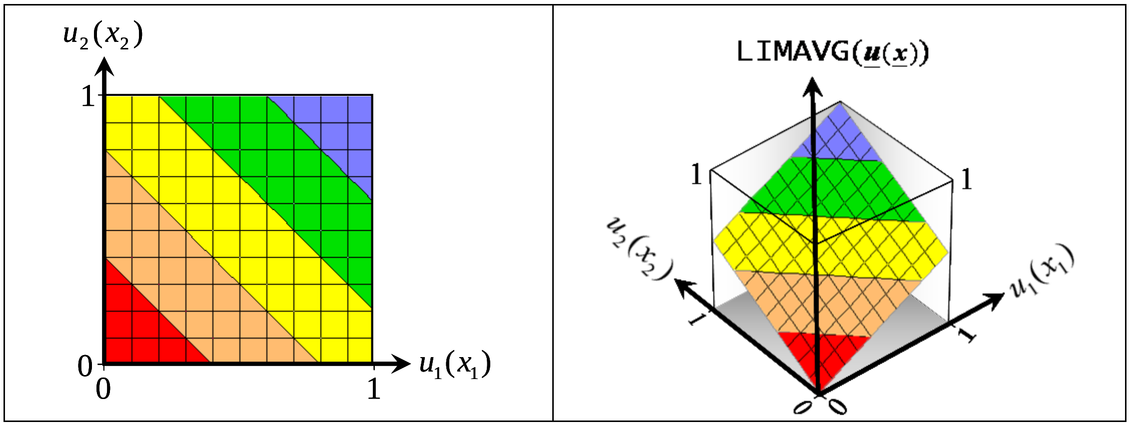
where $\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ and $\mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ are specified in eqs. (1) and (3), respectively, and λ is a parameter. To illustrate the affect of λ, the limited average model was applied to the same project management example described in Section 6. The topographic projections of eq. (41) are shown in Figures 26 through 28. The legend for these figures is summarized in Table 1 in Section 6. Figure 26 shows that when $\lambda = 1$ the limited average model takes the form of the additive model. Figure 27 shows that when $\lambda = 0.2$, the limited average model is a mix of the additive model and the minimization model. Figure 28 shows that when $\lambda = 0$, the limited average model takes the form of the minimization model.

B.2 Exponential Average Model

The exponential average model, denoted $\mathsf{EXPAVG}(\underline{\boldsymbol{u}}(\underline{x}))$, is given by

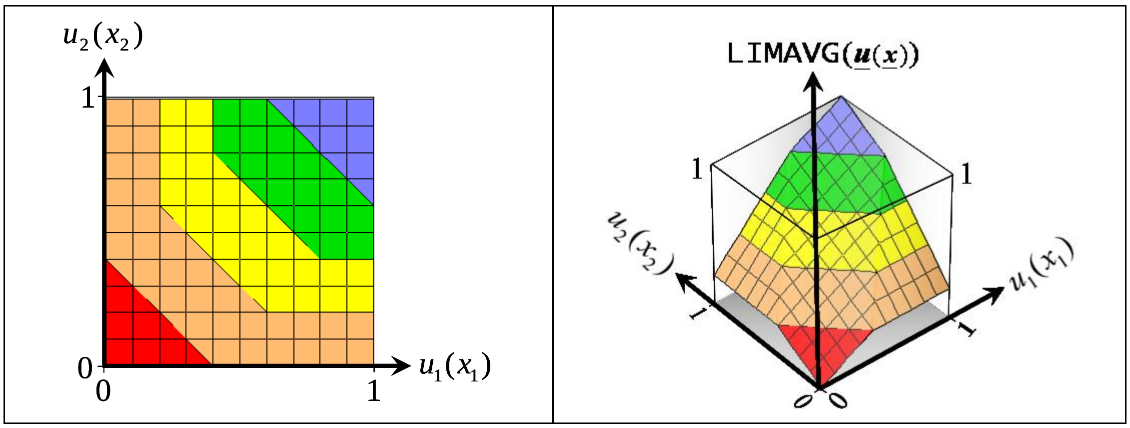
$$\mathsf{EXPAVG}(\underline{\boldsymbol{u}}(\underline{x})) = \log_a \left\{ \sum_{i=1}^{i=n} w_i \cdot a^{u_i(x_i)} \right\} \tag{42}$$

where a is a parameter, $\log_a\{\}$ the base *a* logarithm, the w_i's are a set of *n* non-negative weights (i.e., constants) that sum to unity, and the $u_i(x_i)$'s are a set of *n* single dimensional (i.e., single attribute) utility functions To illustrate the affect of the parameter *a*, the exponential average model was applied to the same project management example described in Section 6. The topographic projections of eq. (42) are shown in Figures 29 through 31. The legend for these figures is summarized in Table 1 in Section 6. Figure 29 shows that when $a = 0.1$ the exponential average model is a mixture of the additive model and the minimization model. Figure 30 shows that as *a* asymptotically approaches one, the exponential average model takes the form of the additive model. Figure 31 shows that when $a = 10$, the exponential average model is a mix of the additive model and a maximization model given by $\max\{u_1(x_1), u_2(x_2), \ldots, u_n(x_n)\}$. In fact, as *a* approaches zero, $\mathsf{EXPAVG}(\underline{\boldsymbol{u}}(\underline{x}))$ approaches a pure minimization model; and as a^{-1} (the reciprocal of *a*) approaches zero, $\mathsf{EXPAVG}(\underline{\boldsymbol{u}}(\underline{x}))$ approaches a pure maximization model. For additional comparisons of the exponential average, the limited average, and the min-additive models, see Schmidt (2007).



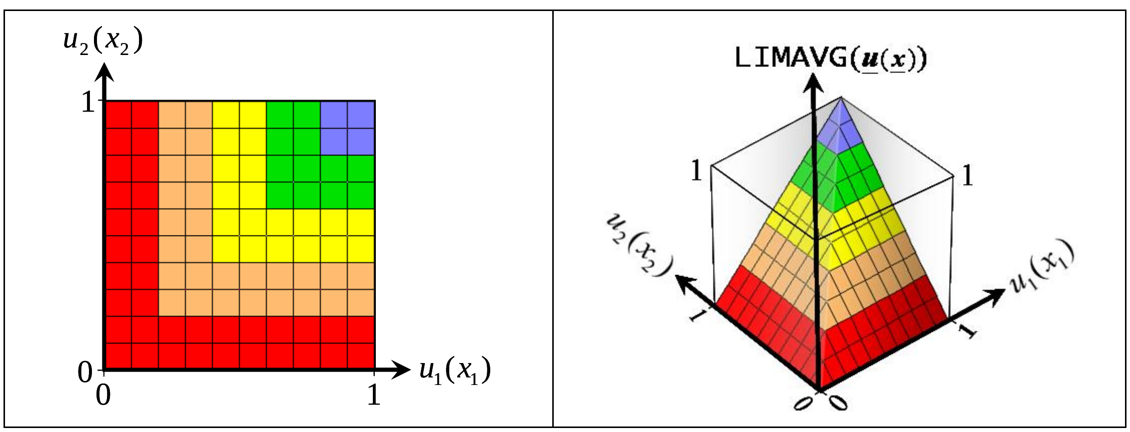
(a) Overhead view (b) Perspective view

Figure 26–Limited average model with $\lambda = 1$



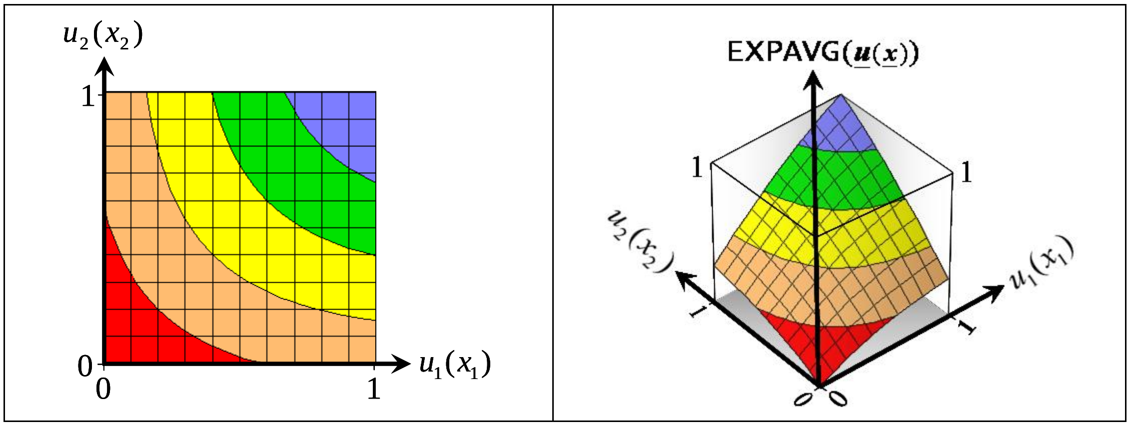
(a) Overhead view (b) Perspective view

Figure 27–Limited average model with $\lambda = 0.2$



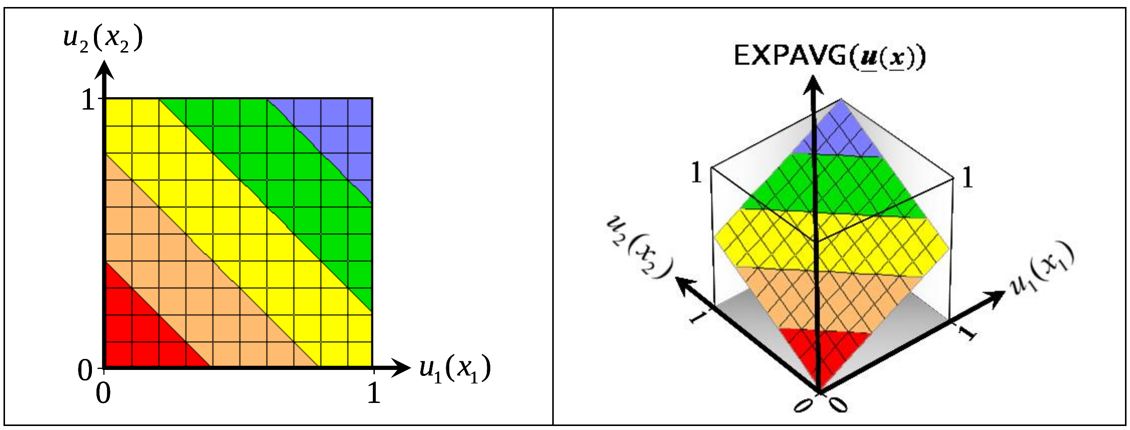
(a) Overhead view (b) Perspective view

Figure 28–Limited average model with $\lambda = 0$



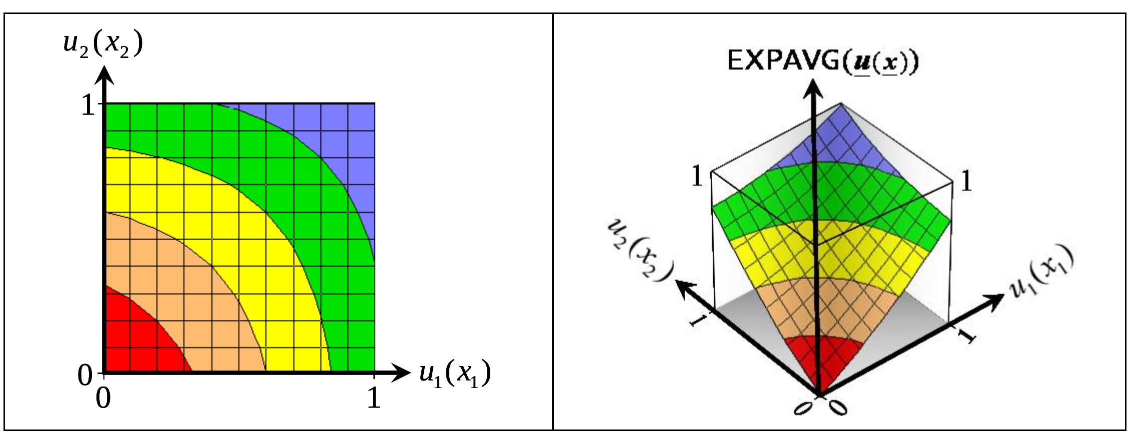
(a) Overhead view (b) Perspective view

Figure 29–Exponential average model with $a = 0.1$



(a) Overhead view (b) Perspective view

Figure 30–Exponential average model with $a \cong 1$



(a) Overhead view (b) Perspective view

Figure 31–Exponential average model with $a = 10$

Appendix C

This appendix summarizes the complement of the min-additive model, namely the *max-additive model*.[8] The max-additive model is intended for decision environments in which the decision maker (DM) wishes to minimize a "disutility function" rather than maximize a utility function. This situation arises frequently in risk analysis where the DM must prepare for a collection of possible outcomes that have undesirable consequences. For example, during the hurricane season, emergency responders must plan for potential high winds, inclement weather, rising water levels, and a host or other uncertain risk events, which, in turn, can trigger other undesirable outcomes. These risk events are disutilities. Typical attributes associated with a risk event are the probability of occurrence; and if the event does occur, the time until the occurrence, the duration of the occurrence, the impact or severity, etc. (see, for example, [Garvey (2000)], [Garvey (2009)]).

A disutility function is the complement of a utility function. That is, low values of the disutility function are more desirable to the DM than high values. Let $d_i(x_i)$ be the ***i***-th single dimensional (i.e., single attribute) disutility function for attribute x_i. For example, x_i might be the number of weeks until a risk event occurs (if it does occur), and $d_i(x_i)$ might be of the form $d_i(x_i) = \exp(-\gamma \cdot x_i)$ where $\exp(\,)$ denotes exponentiation of the Euler number, e, and γ is a time-constant parameter. For low values of this function (when the event, if it occurs, is many weeks away), the DM may be willing to make tradeoffs to mitigate other risk events. On the other hand, if the value of $d_i(x_i)$ is close to one (indicating a possibly imminent event), the DM may be less willing to consider tradeoffs. This asymmetry in the DM's attitude towards tradeoffs is *not* reflected in an additive disutility model. However, it *is* captured in the "max-additive model."

There are four versions of the max-additive model, completely analogous to the four versions of the min-additive model. Let $\mathsf{MAXADD}_k(\underline{\boldsymbol{d}}(\underline{\boldsymbol{x}}))$ denote the *k*-th version of the max-additive model where the index $k = 0$ for the basic version, $k = 1$ for the uniform version, $k = 2$ for the logistic version, and $k = 3$ for the relaxed version; and $\underline{\boldsymbol{d}}(\underline{\boldsymbol{x}}) = \{d_1(x_1), d_2(x_2), \ldots, d_n(x_n)\}$ is a set of *n* single dimensional (i.e., single attribute) disutility functions. Each $d_i(x_i)$ ranges from zero to one (with zero indicating the most preferable value to the DM and 1 indicating the least preferable). The form of the *k*-th version of the max-additive model is given by

$$\mathsf{MAXADD}_k(\underline{\boldsymbol{d}}(\underline{\boldsymbol{x}})) = w_{\mathsf{MA}k}(1-M)\cdot \mathsf{MAX}(\underline{\boldsymbol{d}}(\underline{\boldsymbol{x}})) + \left(1 - w_{\mathsf{MA}k}(1-M)\right)\cdot \mathsf{ADD}(\underline{\boldsymbol{d}}(\underline{\boldsymbol{x}})) \qquad (43)$$

[8] The "max-additive model" is also called the "max-average model." The prefix "max" in the max-additive model is underlined to help to visually distinguish it from the prefix "min" in the min-additive model.

where $\mathsf{ADD}(\underline{\boldsymbol{d}}(\underline{x}))$ is the additive model (see eq. (1)) with the argument $\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})$ replaced by $\underline{\boldsymbol{d}}(\underline{\boldsymbol{x}})$, $w_{\mathsf{MA}_k}(1-M)$ is the k-th min-additive weighting function (see eqs. (12), (14), (19), and (26)), with the argument m replaced with $1-M$, and M and $\mathsf{MAX}(\underline{\boldsymbol{d}}(\underline{x}))$ are defined as

$$M \;=\; \mathsf{MAX}(\underline{\boldsymbol{d}}(\underline{x})) \;=\; \max\{d_1(x_1), d_2(x_2), \ldots, d_n(x_n)\} \tag{44}$$

This structure creates a very straightforward relationship between the corresponding versions of the <u>max</u>-additive and the min-additive models. To specify this relationship, let

$$u_i(x_i) = 1 - d_i(x_i) \qquad \text{for } \; i = 1, \ldots, n \tag{45}$$

Then

$$\mathsf{MAXADD}_k(\underline{\boldsymbol{d}}(\underline{x})) = 1 - \mathsf{MA}_k(\underline{\boldsymbol{u}}(\underline{x})) \qquad \text{for } \; k = 1, \ldots, 4 \tag{46}$$

where $\mathsf{MAXADD}_k(\underline{\boldsymbol{d}}(\underline{x}))$ is defined in eq. (43), $\mathsf{MA}_k(\underline{\boldsymbol{u}}(\underline{x}))$ is defined in eqs. (9), (13), (18), and (27), $\underline{\boldsymbol{d}}(\underline{\boldsymbol{x}}) \;=\; \{d_1(x_1), d_2(x_2), \ldots, d_n(x_n)\}$ is a set of n single dimensional (i.e., single attribute) disutility functions, and $\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}}) \;=\; \{u_1(x_1), u_2(x_2), \ldots, u_n(x_n)\}$. is the complimentary set of n single dimensional utility functions (see eq. (45)).

To prove eq. (46), we note that, by construction, the following relationships hold:

$$\mathsf{MAX}(\underline{\boldsymbol{d}}(\underline{x})) = 1 - \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \tag{47a}$$

$$\mathsf{ADD}(\underline{\boldsymbol{d}}(\underline{x})) = 1 - \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \tag{47b}$$

where $\mathsf{MAX}(\underline{\boldsymbol{d}}(\underline{x}))$ is defined in eq. (44), $\mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ is defined in eq. (3) and $\mathsf{ADD}(\underline{\boldsymbol{d}}(\underline{x}))$ and $\mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x}))$ are defined in eq. (1). Note also that since $m = \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))$ and $M = \mathsf{MAX}(\underline{\boldsymbol{d}}(\underline{x}))$, eq. (47a) can be rewritten as

$$M = 1 - m \tag{48}$$

Substituting eqs.(47) and (48) in eq. (43) yields the following identity:

$$\begin{aligned}
\mathsf{MAXADD}_k(\underline{\boldsymbol{d}}(\underline{x})) &= w_{\mathsf{MA}_k}(1-M) \cdot \mathsf{MAX}(\underline{\boldsymbol{d}}(\underline{x})) \;+\; \left(1 - w_{\mathsf{MA}_k}(1-M)\right) \cdot \mathsf{ADD}(\underline{\boldsymbol{d}}(\underline{x})) \\
&= \left(1 - w_{\mathsf{MA}_k}(m)\right) \cdot \left(1 - \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x}))\right) \;+\; w_{\mathsf{MA}_k}(m) \cdot \left(1 - \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x}))\right) \\
&= 1 - w_{\mathsf{MA}_k}(m) - \left(1 - w_{\mathsf{MA}_k}(m)\right) + w_{\mathsf{MA}_k}(m) \cdot \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x})) \\
&\quad + w_{\mathsf{MA}_k}(m) - w_{\mathsf{MA}_k}(m) \cdot \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x})) \\
&= 1 - \left(\left(1 - w_{\mathsf{MA}_k}(m)\right) \cdot \mathsf{MIN}(\underline{\boldsymbol{u}}(\underline{x})) + w_{\mathsf{MA}_k}(m) \cdot \mathsf{ADD}(\underline{\boldsymbol{u}}(\underline{x}))\right) \\
&= 1 - \mathsf{MA}_k(\underline{\boldsymbol{u}}(\underline{x}))
\end{aligned} \tag{49}$$

Eq. (49) shows that the <u>max</u>-additive model is the complement of the min-additive model and thus shares the properties inherent in the min-additive family of models.

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