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Semigroup structure of singleton Dempster-Shafer evidence accumulation

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Abstract

Dempster-Shafer theory is one of the main tools for reasoning about data obtained from multiple sources, subject to uncertain information. In this work abstract algebraic properties of the Dempster-Shafer set of mass assignments are investigated and compared with the properties of the Bayes set of probabilities. The Bayes set is a special case of the Dempster-Shafer set, where all non-singleton masses are fixed at zero. The language of semigroups is used, as appropriate subsets of the Dempster-Shafer set, including the Bayes set and the singleton Dempster-Shafer set, under either a mild restriction or a slight extension, are semigroups with respect to the Dempster-Shafer evidence combination operation. These two semigroups are shown to be related by a semigroup homomorphism, with elements of the Bayes set acting as images of disjoint subsets of the Dempster-Shafer set. Subsequently, an inverse mapping from the Bayes set onto the set of these subsets is identified and a procedure for computing certain elements of these subsets, acting as subset generators, is obtained. The algebraic relationship between the Dempster-Shafer and Bayes evidence accumulation schemes revealed in the investigation elucidates the role of uncertainty in the Dempster-Shafer theory and enables direct comparison of results of the two analyses.

Keywords: data fusion, evidence accumulation, Dempster-Shafer theory, Dempster-Shafer mass, Bayes inference, uncertainty, semigroup, semigroup homomorphism.

"If we begin with certainties, we shall end in doubts; but if we begin with doubts, and are patient with them, we shall end in certainties."

Francis Bacon, The advancement of learning

"If you demand a rule from which it follows that there can't have been a miscalculation here, the answer is that we did not learn this through a rule, but by learning to calculate."

Ludwig Wittgenstein, On certainty

1 Introduction

Dempster-Shafer (DS) theory is one of the main tools for reasoning about data obtained from multiple sources, subject to uncertain information. The principal task of such reasoning is data fusion, or evidence accumulation. The goals of data fusion is to decrease uncertainty associated with individual measurements and to permit identification of the most likely alternative. The DS approach has applications in several areas, including sensor fusion, medical diagnostics, biometrics, and decision support [2], [12], [13], [18], [19]. A review of some of these applications is given in [3].

Despite the ubiquity of the DS technique in science and engineering, several problems remain unsolved, making an effective translation of theory into practice difficult. Among these problems are: (1) lack of model-based rules for mass assignment, (2) lack of theoretical justification for the evidence combination rule, (3) lack of an appropriate formalism that would allow one to interpret evidence combination results in probabilistic terms, (4) high asymptotic complexity of the evidence combination computation, and (5) unsatisfactory treatment of incompatible evidence [9], [8], [25], [24].

Many alternative approaches to evidence accumulation have been proposed to address these issues. They include: the transferable belief model (TBM) of Smets [5], the modified DS approach (MDS) of Fixsen and Mahler [16], the conditioned DS theory of Mahler [17], the connectionist-based neural network approach of Basir *et al* [1], the proportional sum approach of Shi *et al* [22], the composition of dependence method of Wu *et al* [23], and the fuzzy-value of measure approach of Lucas and Araabi [15]. While these approaches seem to address some of the aforementioned concerns to a certain degree, they often increase the complexity of the analysis and generally yield different results. Since no consensus on which of these methods is to be preferred has emerged, it appears that, despite its shortcomings, the original DS approach still remains the standard reference.

In this work we focus on the relationship between the DS and Bayesian evidence accumulations. There are several reasons for that. First, this relationship encapsulates the key ideas of data fusion and links all of the aforementioned problems. Second, since the latter technique is both more familiar and better understood, it is reasonable to expect that specification of the relationship between the two analyses will produce new insights into the DS theory. While it is accepted that the DS theory is, in a certain sense, a generalization of the probability theory [2], the approaches vary in several important respects, including the treatment of uncertain information and the way the evidence is combined, making direct comparison of results of the two analyses difficult. In this work we ameliorate these difficulties by proposing a mathematical framework within which the relationship between the two methods can be made precise. The findings of the investigation elucidate the role uncertainty plays in the DS theory and enable evaluation of relative fitness of the two techniques for practical data fusion scenarios.

The approach chosen for the analysis of the DS evidence accumulation is based on semigroup theory. This is appropriate as certain subsets of the DS set are semigroups with respect to the DS evidence combining operation. The abstract algebra approach allows one to access the DS theory at the most general level, highlighting its most essential properties. Focusing on the semigroup-theoretic structures of the fundamental DS concepts reveals key relationships between certain special cases of the DS analyses and, most importantly, between the DS and Bayes analyses.

Throughout this work we will find it convenient to distinguish between several special types of the DS mass assignment. We will refer to the set of DS mass with arbitrary uncertainty value as the *DS set* and to the set with zero uncertainty as the *Bayes set*. Similarly, the analyses performed on these two sets will be called the DS and Bayes analyses, respectively. Furthermore, we will identify two cases of DS analysis: the *singleton* DS analysis and the *general*, or *full* DS analysis. In the former case the set of evidence includes only the singletons and the universal set, the *uncertainty*. In the latter case the set of evidence includes all non-empty subsets of the universal set. To make presentation concise, only the singleton set is being considered in this paper. We remark, however, that the key ideas and results described here apply to the general case as well. Analysis of the full DS formalism will be given elsewhere [7]. The singleton case is important, as it succinctly captures the main features of the DS theory, and as it is often used, due to its reduced complexity, in practical data fusion systems.

The analysis begins with identification of two subsets of the DS set for which the combining operation is well-defined. It is shown that these sets, together with the set of anomalous mass assignments, form a partition of the singleton DS set. As the former two sets, together with the evidence combining operation, are semigroups, the evidence admissibility condition for the evidence combining operation can be conveniently replaced by the closure condition for a semigroup, and the DS analysis can be replaced by a semigroup-theoretical analysis on subsets of the DS set. This leads to two results. First, it is shown that the direct sum of the two semigroups is a semigroup, and that it is the largest semigroup contained in the singleton DS set. This result is satisfactory, as it excludes only the set of anomalous mass assignments. Second, a homomorphism from the direct sum of the two principal semigroups onto one of the individual semigroups is specified. This homomorphism reduces the analysis on the DS set to the analysis on a subset of the DS set that is identical with the set of non-zero Bayes probabilities, and replaces the complex evidence combining operation with a pointwise multiplication. A special role is played by the pre-image of the identity element of the Bayes set, given by a subset of the DS set made up of elements of equal mass, except for the mass of uncertainty. This pre-image coincides with the kernel of the homomorphism. The homomorphism induces an equivalence relation on the DS set, which leads to the construction of a factor group of the DS set with respect to the kernel of the homomorphism. The DS set is thus partitioned into disjoint subsets (called equivalence classes, or pseudocosets) that are associated with the elements of the factor set; these subsets, in turn, by a group isomorphism, can be unambiguously identified with the elements of the Bayes set. In effect, combination of DS elements associated with two pseudocosets can be replaced with combination of corresponding Bayes elements. As the fundamental relationship between the DS factor set and the Bayes set is a group isomorphism, certain elements of pseudocosets act as pseudocoset generators. These generators are identified and shown to be the elements of pseudocosets that have the largest uncertainty. Among the main results, the expression for the pseudocoset generator demonstrates that the structure of a pseudocoset is fully characterized by its Bayesian image and the value of uncertainty. The investigation concludes with extension of the semigroup formalism to the unrestricted singleton DS set.

The content of the paper is as follows. Section 2 reviews the DS theory. Section 3 provides a resume of some elementary facts of group and semigroup theory. Section 4 investigates the semigroup structure of the restricted singleton DS set and states the main results. Section 5 extends the analysis to the unrestricted singleton DS set. Section 6 gives concluding remarks.

2 Elements of the DS theory

Denote by Ω a finite non-empty set of all possible outcomes of an event of interest, and by 2^Ω the power set of Ω. Define the set of observable outcomes, called the *set* (of subsets) *of evidence*, by

$$A = \{A_i \mid 0 < i \leq |A|\} \subseteq 2^\Omega, \quad A \neq \varnothing, \tag{1}$$

where $|A|$ is the cardinality of A, and $\subseteq$ denotes "is a subset of".

Given the set A in (1), define a mapping

$$m_A : A \mapsto [0, 1], \tag{2}$$

such that

$$m_A(\varnothing) = 0 \tag{3}$$

and

$$\sum m_A(A_i) = 1. \tag{4}$$

Set $a_i = m_A(A_i)$ and call it the *mass* of A_i. By an abuse of notation we will also write

$$a = \{a_i \mid 0 < i \leq |A|\}, \tag{5}$$

and refer to a as the *mass assignment* of A. Finally, we will call the set of pairs of the subsets A_i and the corresponding masses a_i,

$$\mathcal{A} = \{(A_i, a_i) \mid 0 < i \leq |A|\}, \tag{6}$$

the *body of evidence* of A. We will distinguish a special case of (6),

$$\mathcal{A} = \{(A_i, a_i) \mid 0 < i \leq n+1\}, \tag{7}$$

where

$$A_i \in \Omega \text{ when } i \leq n,\ A_{n+1} = \Omega \text{ and } |\Omega| = n. \tag{8}$$

We will call the body of evidence in (7) the singleton body of evidence.

The key difference between probability and mass is that probability is a measure and therefore it satisfies the additivity condition, that is, given a finite sequence A_i, $0 < i \leq |A|$, of disjoint subsets of A,

$$P\left(\bigcup A_i\right) = \sum P(A_i). \tag{9}$$

In general, the condition (9) is not satisfied by mass. Removing the additivity constraint can be convenient, as it permits inclusion of subjective judgments in the DS information fusion system, but it also has the undesirable consequence of making the interpretation of results of such fusion problematic. In particular, when considered together with the DS rule of combination, it is not always clear when mass can be made consistent with the standard probability evaluation.

A key feature of the DS theory is the rule for combining bodies of evidence. Let $\mathcal{A}$ and $\mathcal{B}$ be two distinct bodies of evidence, given by (6). The DS rule for combining the masses of $\mathcal{A}$ and $\mathcal{B}$ is then

$$c_k = \frac{1}{1-\kappa} \sum_{A_i \cap B_j = C_k} a_i b_j, \quad 0 < k \leq |C|, \tag{10}$$

where

$$\kappa = \sum_{A_i \cap B_j = \varnothing} a_i b_j \neq 1 \tag{11}$$

is the *conflict coefficient* and

$$\mathcal{C} = \{(C_k, c_k) \mid 0 < k \leq |C|\} \tag{12}$$

is the DS composite body of evidence.

Apart from mass, two other concepts play a key role in the DS theory: balance and plausibility. *Balance* (also, *belief* or *support*) of a subset A_i is the sum of the masses of all subsets A_j of A, that are also subsets of A_i, i.e.,

$$b_{A_i} = \sum_{A_j \subseteq A_i} a_j, \quad 0 < i \leq |A|. \tag{13}$$

Plausibility of a subset A_i is the sum of the masses of all subsets A_j of A, having non-empty intersection with A_i, i.e.,

$$p_{A_i} = \sum_{A_i \cap A_j \neq \varnothing} a_j, \quad 0 < i \leq |A|. \tag{14}$$

Like mass, balance and plausibility are mappings from the power set of Ω to the unit interval. In particular,

$$b_{\varnothing} = p_{\varnothing} = 0 \tag{15}$$

and

$$b_{\Omega} = p_{\Omega} = 1. \tag{16}$$

Balance and plausibility are related by the formula

$$p_{A_i} = 1 - b_{\bar{A}_i}, \quad 0 < i \leq |A|, \quad \bar{A}_i = \Omega - A_i. \tag{17}$$

Using Rota's generalization of the Möbius inversion theorem [20], mass can be uniquely recovered from balance by the formula

$$a_j = \sum_{A_i \subseteq A_j} (-1)^{|A_j - A_i|} b_{A_i}, \quad 0 < j \leq |A|. \tag{18}$$

A key result in DS theory describes the relationship between balance, plausibility and probability. It follows from (13) and (14) that

$$b_{A_i} \leq p_{A_i}, \quad 0 < i \leq |A|. \tag{19}$$

A stronger version of (19), that allows comparison of results of DS and probabilistic analyses, is given by

$$b_{A_i} \leq P(A_i) \leq p_{A_i}, \quad 0 < i \leq |A|. \tag{20}$$

(20) has been proposed by Dempster [6], for the situation where mass assignment arises from a set-valued mapping from a probability space to Ω. Since balance and plausibility bound the value of probability in (20), they are often referred to as the *lower* and *upper probabilities*.

3 Some basic facts of abstract algebra

The main algebraic structures considered in this paper are abelian semigroups, monoids and groups [4], [10], [11]. We will review the properties of these structures that are necessary for our constructions.

Suppose S is a non-empty set and $\circ$ is a binary operation on S. If $\circ$ is *associative*, i.e.,

$$a \circ (b \circ c) = (a \circ b) \circ c \quad \text{for all} \quad a, b, c \in S,$$

then S is a *semigroup*. If, additionally, $\circ$ is *commutative*, i.e.,

$$a \circ b = b \circ a \quad \text{for all} \quad a, b \in S,$$

then the semigroup S is an *abelian* semigroup. A semigroup S that contains the *identity* element e, i.e.,

$$a \circ e = e \circ a = a \quad \text{for all} \quad a \in S,$$

is a *monoid*. A monoid S such that for every $a \in S$ there is an *inverse* element $a^{-1} \in S$, i.e.,

$$a \circ a^{-1} = a^{-1} \circ a = e \quad \text{for all} \quad a \in S,$$

is a *group*. While there are more general structures than semigroups, called *groupoids* or *magmas*, that do not require associativity, they are of limited use, as the product $a \circ b \circ c$ in these cases is not unique. However, we find it convenient, for the purpose of this work, to employ the concept of *partial semigroup*. Partial semigroup is a non-empty set with associative binary operation defined on *some* pairs of its elements. The algebra of partial operations, including partial semigroups, is described in detail in [14].

Suppose S is a semigroup and T is a nonempty subset of S. If T is a semigroup under the operation in S, then T is a *subsemigroup* of S, denoted $T < S$. The same convention is used for monoids and groups.

Of fundamental importance in theory and applications are mappings between sets that preserve algebraic structures. One of these mappings is a semigroup (monoid, group) homomorphism.

Suppose $(S, \circ)$ and $(T, \star)$ are semigroups. A mapping $\phi : S \to T$ is a *semigroup homomorphism* iff

$$\phi(a \circ b) = \phi(a) \star \phi(b) \quad \text{for all} \quad a, b \in S.$$

If $(S, \circ)$ and $(T, \star)$ are monoids with the identity elements e_S and e_T, respectively, the semigroup homomorphism $\phi : S \to T$ is a *monoid homomorphism* iff

$$\phi(e_S) = e_T.$$

Group homomorphism is defined identically to semigroup homomorphism, since for groups conservation of identity elements follows from conservation of group operation. Several special types of homomorphism (regardless of structure) need to be mentioned. If ϕ is one-to-one, ϕ is a monomorphism. If ϕ is onto, ϕ is an epimorphism.[1] If ϕ is one-to-one and onto, ϕ is an isomorphism.

Let $\phi : S \to T$ be a monoid epimorphism. ϕ induces an equivalence relation $ker(\phi)$ on S,

$$ker(\phi) = \{(a, b) \in S \times S \mid \phi(a) = \phi(b)\},$$

called the *kernel* of ϕ. Define the *factor monoid*, $S/ker(\phi)$, as the set of all equivalence classes,

$$[c] = \{a \in S \mid \phi(a) = c\}, \; c \in T,$$

and a mapping, $\chi : S \to S/ker(\phi)$, that sends all elements in S to their equivalence classes. The mapping χ is called the *canonical epimorphism* or *projection*. Factor sets can be alternatively described as partitions. A partition π of a set S is a set π whose elements are subsets of S such that each $a \in S$ is an element of exactly one subset. The monoid epimorphism ϕ induces a monoid isomorphism, $\psi : S/ker(\phi) \to T$, given by $\psi([c]) = c$. This yields the relationship $\phi = \psi \diamond \chi$, where $\diamond$ denotes composition of mappings. This relationship can be expressed in a slightly more general form when the homomorphism ϕ is not onto.

When S and T are groups and ϕ is a group homomorphism, then the kernel of ϕ, now labeled K, describes the equivalence class of the identity element of T,

$$K = [e_T] = \{a \in S \mid \phi(a) = e_T\}.$$

[1] Note, that we are using an "algebraic" definition of epimorphism. In category theory epimorphism is defined more broadly, by its right-cancellative property [10].

The group epimorphism ϕ induces a group isomorphism, $\psi : S/K \to T$. Unlike in monoid homomorphism, in group homomorphism the equivalence class of e fully characterizes the structure of the factor set. This is because in a group a relation $a = b$ is equivalent to $ab^{-1} = e$, and hence to study relations satisfied in the image of ϕ it is sufficient to consider the equivalence class of e. The elements of the factor group S/K are called the *cosets* of K in S. The cosets of K in S have a special structure: they are explicitly given by the sets

$$aK = \{ab \mid b \in K\},\ a \in S.$$

The above construction for cosets does not hold, in general, for monoid equivalence classes, and further investigation is usually required to determine their composition.

The structures considered in this paper are of mixed type: the domain of the monoid epimorphism, ϕ, is a monoid, S, while the co-domain is a group, T. In this case, the factor set in S that is isomorphic to the image of ϕ is the factor group S/K, and the equivalence relation $ker(\phi)$ is completely determined by the equivalence class of e_T, K. Since S/K is a group, equivalence classes of the monoid epimorphism ϕ resemble group-theoretical cosets, in that each equivalence class can be obtained by a composition of K with some element of S. However, since S and K are not groups, not every element in S generates an equivalence class. To reflect these mixed properties, we will refer to the elements of S/K in this case as *pseudocosets*.

Two concepts that play a key role in the theory of semigroups are idempotents and ideals. Let S be a semigroup and $A \subset S$. If

$$AS = A,$$

where $AS = \{as \mid a \in A \text{ and } s \in S\}$, then A is an *ideal* of S. The minimal ideal of an abelian semigroup, when it exists, is a group. Ideals are instrumental in the construction of a certain type of equivalence relations called Green equivalence relations, which are fundamental to several important semigroup classes. In this work the ideal property of the homomorphic image with respect to the domain underlines the special relationship between the DS and Bayes sets.

An element $a \in S$ is an *idempotent* iff $a \circ a = a$. Each group (and therefore each subgroup) contains one idempotent, the identity element. Semigroups may contain multiple idempotents.

Idempotents determine, in part, subgroups of a semigroup, the cancellative properties of a semigroup, and the existence of an embedding of a semigroup in a group. The last property in particular is relevant here, as semigroups with more than one idempotent cannot be embedded in a group, which is true about the DS semigroups considered in the next two sections, $S_0 \oplus S_1$ and S, and hence a group-theoretical analysis cannot be performed on an algebraic extension of the DS set.

We conclude with the remark that while *sensu stricto* the subjects of this work are monoids, since semigroups can always be trivially extended to monoids, and since monoid theory is an integral part of semigroup theory, in statements of results and in discussions we will generally refer to monoids as semigroups.

4 Semigroup structure of the restricted DS set of singleton mass

In this section we identify the DS set of singleton mass, S, and investigate algebraic properties of certain subsets of S. In particular, we focus on the admissibility condition for the DS combination operation. The investigation reveals semigroup structure of the DS set that has several implications for practical implementations of DS evidence accumulation, including the design of fast algorithms. We conclude with a result relating the singleton DS analysis with the Bayes analysis.

4.1 Partition of S

Take S to be an infinite set of $(n+1)$-tuples

$$a = (a_1, ..., a_{n+1}), \quad 0 \leq a_k \leq 1, \quad \sum a_k = 1. \tag{21}$$

In the DS theory a is called the mass assignment of A and a_k is called the mass of A_k. Suppose S is the set of mass assignments associated with singleton evidence, given by (7)-(8), and

$a, b \in S$. Then the binary evidence combining operation $\circ$ on S is given by

$$c_k = (a \circ b)_k \doteq \begin{cases} \frac{a_k b_k + a_k b_{n+1} + a_{n+1} b_k}{1-\kappa}, & 1 \le k \le n, \\ \\ \frac{a_{n+1} b_{n+1}}{1-\kappa}, & k = n+1, \end{cases} \tag{22}$$

where

$$\begin{aligned} \kappa &= \sum_{s \neq n+1} a_s \sum_{l \neq s, n+1} b_l \\ &= (1 - a_{n+1})(1 - b_{n+1}) - < a, b > + a_{n+1} b_{n+1} \\ &= 1 - < a, b > - a_{n+1} - b_{n+1} + 2 a_{n+1} b_{n+1}. \end{aligned} \tag{23}$$

For $\circ$ to be well-defined, we need

$$1 - \kappa = < a, b > + a_{n+1} + b_{n+1} - 2 a_{n+1} b_{n+1} \neq 0. \tag{24}$$

The condition (24) cannot be satisfied for all $a, b \in S$. However, we can identify subsets of S,

$$S_0 \doteq \left\{ (a_1, ..., a_{n+1}) \;\middle|\; \sum a_k = 1, \quad 0 < a_k < 1 \;\text{ when }\; k \le n \;\text{ and }\; a_{n+1} = 0 \right\} \tag{25}$$

and

$$S_1 \doteq \left\{ (a_1, ..., a_{n+1}) \;\middle|\; \sum a_k = 1, \quad 0 \le a_k < 1 \;\text{ when }\; k \le n \;\text{ and }\; 0 < a_{n+1} \le 1 \right\}, \tag{26}$$

for which the condition (24) is met. The first set consists of non-zero mass assignments to all singletons and to no other set. The second set includes all mass assignments with non-zero uncertainty. In the former case the condition (24) reduces to the condition

$$1 - \kappa = < a, b > \neq 0 \;\text{ iff }\; a_k b_k \neq 0 \;\text{ for some }\; k \neq n+1. \tag{27}$$

The condition (27) is trivially satisfied, since, by the restriction of S_0, $a_k > 0$ for $k \le n$. In the latter case, by the restriction of S_1, we have

$$< a, b > - a_{n+1} b_{n+1} \ge 0 \tag{28}$$

and

$$a_{n+1} + b_{n+1} - a_{n+1} b_{n+1} > 0, \tag{29}$$

and therefore, again, the condition (24) is satisfied.

To complete the analysis, consider the set

$$S_0' \doteq S - S_0 \cup S_1 = \left\{ (a_1, ..., a_{n+1}) \;\middle|\; \sum a_k = 1, \quad \prod_{k \neq n+1} a_k = 0, \quad a_{n+1} = 0 \right\}. \tag{30}$$

Since there are many pairs of elements in S_0', for which $\kappa = 1$, the binary operation $\circ$ is, in general, not well-defined on S_0'. $(S_0', \circ)$ and $(S, \circ)$ are partial semigroups. S_0' can be viewed as a generalization of the Zadeh set of anomalous mass assignments [25].

Since

$$S = S_0 \cup S_0' \cup S_1 \tag{31}$$

and

$$S_0 \cap S_0' = S_0 \cap S_1 = S_0' \cap S_1 = \varnothing, \tag{32}$$

S is a direct sum of S_0, S_0' and S_1, i.e.,

$$S = S_0 \oplus S_0' \oplus S_1. \tag{33}$$

In the remainder of the paper we will investigate abstract properties of S_0, S_1 and S_0'. This investigation will lead to the construction of a general framework for computing in S. In particular, in section 5, we address the DS computation in S_0'.

Although we distinguish between the restricted singleton DS set, $S_0 \oplus S_1$, and the unrestricted singleton DS set, $S \oplus \mathbf{0}$, the latter introduced in section 5, for brevity we will often refer to both simply as singleton DS sets, when no ambiguity arises, as the results of section 5 mirror the results of section 4. Similarly, we will refer to both, S_0 and $S_0 \oplus S_0' \oplus \mathbf{0}$, as Bayes sets.

All sets considered in this work are associative and commutative with respect to $\circ$. We conclude with the demonstration of associativity of $(S_0 \oplus S_1, \circ)$. Proof of commutativity follows directly from commutativity of products of reals.

Lemma 1 $\circ$ *is associative on* $S_0 \oplus S_1$.

Proof Take $a, b, c \in S_0 \oplus S_1$ and fix $\lambda = 1 - \kappa$. Then, by associativity of products of reals and intersections of sets,

$$\begin{aligned}(a \circ b) \circ c &= \frac{1}{\lambda_1} \sum_{X_i \cap C_j = Y_m} \left(\frac{1}{\lambda_2} \sum_{A_k \cap B_l = X_i} a_k b_l \right) c_j \\ &= \frac{1}{\lambda_1 \lambda_2} \sum_{(A_k \cap B_l) \cap C_j = Y_m} (a_k b_l) c_j, \quad 1 \leq m \leq n+1, \\ &= a \circ (b \circ c),\end{aligned}$$

where

$$\lambda_1 = \sum_m \sum_{X_i \cap C_j = Y_m} x_i c_j, \quad \lambda_2 = \sum_i \sum_{A_k \cap B_l = X_i} a_k b_l,$$

and

$$\lambda_1 \lambda_2 = \sum_m \sum_{(A_k \cap B_l) \cap C_j = Y_m} (a_k b_l) c_j = \sum_m \sum_{A_k \cap (B_l \cap C_j) = Y_m} a_k (b_l c_j). \ \square$$

Associativity of $(S_0', \circ)$ follows in cases when the DS combination is allowed.

4.2 Computing in S_0

Consider S_0, the set of $(n+1)$-tuples, as in equation (25),

$$a = (a_1, ..., a_n, 0), \quad 0 < a_k < 1, \quad k \leq n, \quad \sum a_k = 1. \tag{34}$$

For $a, b \in S_0$ the binary operation $\circ$ reduces to

$$c_k = (a \circ b)_k = \begin{cases} \frac{a_k b_k}{1-\kappa_0}, & 1 \leq k \leq n, \\ 0, & k = n+1, \end{cases} \tag{35}$$

where

$$1 - \kappa_0 = 1 - \sum a_s \sum_{l \neq s} b_l =< a, b > . \tag{36}$$

While the equations (35)-(36) are convenient in computations, for generality throughout this section $\circ$ will denote the operation given by (22).

Example 1 Take $a = b = (\alpha, \beta, \gamma, 0)$. Then, by (35),

$$c = \left(\frac{\alpha^2}{\alpha^2 + \beta^2 + \gamma^2}, \frac{\beta^2}{\alpha^2 + \beta^2 + \gamma^2}, \frac{\gamma^2}{\alpha^2 + \beta^2 + \gamma^2}, 0 \right).$$

First, we will check if S_0 has an identity element, i.e., if there is $a \in S_0$, such that

$$a \circ b = b \circ a = b \quad \text{for} \quad \text{all} \quad b \in S_0. \tag{37}$$

When the condition (37) is satisfied, we will write $a = 1_{S_0}$. Suppose $a = 1_{S_0}$. Then, by (36),

$$\begin{aligned} a_k &= 1 - \sum a_s \sum_{l \neq s} b_l \\ &= 1 - \sum a_s (1 - b_s) \\ &= \sum a_s b_s, \quad k \neq n+1. \end{aligned} \tag{38}$$

The last equality is true iff $a_k = \frac{1}{n}$ for all $k \leq n$. In effect,

$$1_{S_0} = e_0 \doteq \left(\frac{1}{n}, ..., \frac{1}{n}, 0 \right). \tag{39}$$

The inverse of a, a^{-1}, is given by

$$(a^{-1})_k = \frac{\prod_{j \neq k} a_j}{\sum_i \prod_{j \neq i} a_j} = \frac{a_k^{-1}}{\sum_i a_i^{-1}}, \quad i, j, k \neq n+1. \tag{40}$$

To verify validity of (40), note that the conditions

$$\sum a_k^{-1} = 1 \quad \text{and} \quad a_1 a_1^{-1} = ... = a_n a_n^{-1} \tag{41}$$

are satisfied. Since closure and associativity hold as well, we have the following result.

Lemma 2 $(S_0, \circ)$ *is a group with the identity* e_0*.*

Example 2 Take $a = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0\right)$. Then, by (40), $a^{-1} = \left(\frac{2}{11}, \frac{3}{11}, \frac{6}{11}, 0\right)$.

4.3 Computing in S_1

Consider the set S_1 with the binary operation $\circ$, given by (22). It follows from previous discussion that S_1 is a semigroup. We want to find if S_1 is a group. Suppose there is $a \in S_1$, such that $a \circ b = b \circ a = b$ for all $b \in S_1$. Then by (22) and (23)

$$a_k = \frac{b_k(1-\kappa-a_{n+1})}{b_k+b_{n+1}}, \quad \text{for} \quad k \leq n, \tag{42}$$

and

$$a_{n+1} = 1-\kappa. \tag{43}$$

It follows that the identity of S_1 is

$$1_{S_1} = e_1 \doteq (0, ..., 0, 1) \, . \tag{44}$$

Moreover, it follows from (26) and (22) that, except for e_1, no element in S_1 has an inverse.[2] This leads to the following result:

Lemma 3 $(S_1, \circ)$ *is a monoid with the identity* e_1*.*

4.4 Computing in $S_0 \oplus S_1$

From previous discussion we know how to compute in S_0 and S_1. Next, we investigate the algebraic structure of $S_0 \oplus S_1$. Consider the combination of $a \in S_0$ and $b \in S_1$ under the operation $\circ$, given by (22). To check if $\circ$ is well-defined in $S_0 \oplus S_1$, observe that, since $b_{n+1} \neq 0$, then

$$1-\kappa = < a, b > + b_{n+1} \neq 0. \tag{45}$$

Moreover, since

$$< a, b > + b_{n+1} = < a, b + b_{n+1} >, \tag{46}$$

[2]In DS calculus, lack of inverses follows from the uncertainty reduction formula, given by theorem 4 in section 5.

then

$$c_k = (a \circ b)_k = \begin{cases} \frac{a_k(b_k+b_{n+1})}{<a,b+b_{n+1}>}, & 1 \le k \le n, \\ \\ 0, & k = n+1. \end{cases} \tag{47}$$

It follows that $0 < c_k < 1$ for $k \le n$ and $c_{n+1} = 0$, and therefore

$$S_0 \circ (S_0 \oplus S_1) = S_0. \tag{48}$$

Since $\circ$ is well-defined on $S_0 \oplus S_1$, $(S_0 \oplus S_1, \circ)$ is a semigroup, and, by (48), S_0 is an *ideal* of $S_0 \oplus S_1$. Moreover, since $e_1 = 1_{S_1}$ and, for any $a \in S_0$,

$$e_1 \circ a = a \circ e_1 = a, \tag{49}$$

then

$$e_1 = 1_{S_0 \oplus S_1}. \tag{50}$$

In general, the elements in $S_0 \oplus S_1$ do not have inverses, and therefore $S_0 \oplus S_1$ is not a group. Hence, we have the following result.

Lemma 4 $(S_0 \oplus S_1, \circ)$ *is a monoid with the identity* e_1.

Of special interest is the composition of an element in $S_0 \oplus S_1$ with e_0. Take $a \in S_0 \oplus S_1$ and define the mapping

$$\phi : a_k \mapsto b_k = a_k \circ e_0. \tag{51}$$

By (47),

$$a_k \circ e_0 = \begin{cases} \frac{a_k + a_{n+1}}{1+(n-1)a_{n+1}}, & 1 \le k \le n, \\ \\ 0, & k = n+1. \end{cases} \tag{52}$$

The next property is the main result of this paper.

Theorem 1 *ϕ is a monoid homomorphism from $S_0 \oplus S_1$ onto S_0.*

Proof Since $\phi(S_0) = S_0$ and $\phi(S_1) \subseteq S_0$,[3] ϕ is onto. To prove that ϕ is a monoid homomorphism, we need to establish that

$$\phi(a \circ b) = \phi(a) \circ \phi(b), \quad a, b \in S_0 \oplus S_1,$$

and

$$\phi(e_1) = e_0.$$

It follows from associativity and commutativity of $(S_0 \oplus S_1, \circ)$ that

$$\phi(a \circ b) = (a \circ b) \circ e_0 = (a \circ b) \circ (e_0 \circ e_0) = (a \circ e_0) \circ (b \circ e_0) = \phi(a) \circ \phi(b).$$

The second condition follows from the fact that ϕ is an epimorphism. □

The homomorphism ϕ induces an equivalence relation, $ker(\phi)$, on $S_0 \oplus S_1$,

$$ker(\phi) = \{(a, b) \in (S_0 \oplus S_1) \times (S_0 \oplus S_1) \mid \phi(a) = \phi(b)\}. \tag{53}$$

The image of the homomorphism, S_0, is isomorphic with the factor group, $S_0 \oplus S_1/ker(\phi)$, the collection of all inverse images of S_0 under ϕ, formally $\psi : S_0 \oplus S_1/ker(\phi) \to S_0$, and the factor group $S_0 \oplus S_1/ker(\phi)$ is a homomorphic image of $S_0 \oplus S_1$, formally $\chi : S_0 \oplus S_1 \to S_0 \oplus S_1/ker(\phi)$. The inverse images of S_0 under ψ form equivalence classes under $ker(\phi)$.

Consider a subset of $S_0 \oplus S_1$,

$$K = \left\{ (a_1, ..., a_{n+1}) \;\middle|\; a_k = \frac{1 - a_{n+1}}{n}, \quad k \neq n + 1, \quad \text{and} \quad 0 \leq a_{n+1} \leq 1 \right\}. \tag{54}$$

Then,

$$\text{for all } a \in K, \quad \phi(a) = e_0 = 1_{\phi(S_0 \oplus S_1)}. \tag{55}$$

In fact, since for no other elements of $S_0 \oplus S_1$ the condition (55) is satisfied, K is the pre-image of ϕ. It is not difficult to see that K is a monoid with the identity element

$$1_K = e_1. \tag{56}$$

[3]Subsequently it will be shown that $\phi(S_1) = S_0$.

It follows that K is the equivalence class of e_0. In general, an equivalence class of an element $b \in S_0$ is the set

$$[b] = \{a \in S_0 \oplus S_1 \mid \phi(a) = b\}. \tag{57}$$

Since $S_0 \oplus S_1/ker(\phi)$ is a group, K fully characterizes the structure of $S_0 \oplus S_1/ker(\phi)$, and hence we can write $S_0 \oplus S_1/ker(\phi) = S_0 \oplus S_1/K$.

In the remainder of the section we will investigate the relationship between equivalence classes of $S_0 \oplus S_1$. This relationship is somewhat more complicated here than when the underlying structure is a group. An explicit construction of equivalence classes of $S_0 \oplus S_1$, which delineates this relationship, is guided by the next two results.

Lemma 5 *Take $a \in S_0 \oplus S_1$ and let ϕ be the monoid homomorphism in (51), with $\phi(a) = b$. ϕ preserves magnitude ordering among components of a and b, i.e., $a_p < a_q$ implies $b_p < b_q$ for all $p, q \neq n+1$.*
Proof Follows directly from (52).

Theorem 2 *Denote an arbitrary equivalence class of ϕ in $S_0 \oplus S_1$ by $[b]$, and its image under ϕ by b. Suppose b_k, $k \neq n+1$, is the smallest component of b, except for b_{n+1}. The following conditions are satisfied.*

1. *Each equivalence class $[b]$ contains a unique element $a^* \in S_1$, such that among all elements of $[b]$, a^* has the largest value of a_{n+1}.*

2. *a^* is given by*

$$a_i^* = \begin{cases} \frac{b_i - b_k}{1-(n-1)b_k}, & i = 1, 2, ..., n, \\ \\ \frac{b_k}{1-(n-1)b_k}, & i = n+1. \end{cases} \tag{58}$$

3. *a^* generates all elements in $[b]$, by the composition a^*K, and it is the only element in $S_0 \oplus S_1$ to do so.*

Proof Take an arbitrary element $b \in S_0$. Suppose b_k, $k \neq n+1$, is the smallest component of b, except for b_{n+1}. Then, by lemma 5, each element a in the inverse image of ϕ, $[b]$, has a corresponding smallest component a_k. If a is distinct from b, then $a_i < b_i$ for all $i \neq n+1$. In particular, $[b]$ admits an element a^*, such that $a_k^* = 0$. Then, by (52),

$$b_k = \frac{a_{n+1}^*}{1+(n-1)a_{n+1}^*} = \frac{a_k' + a_{n+1}'}{1+(n-1)a_{n+1}'}$$

for all $a' \in [b]\backslash\{b\}$. It follows that either $a_k' = 0$ and $a_{n+1}' = a_{n+1}^*$, and hence $a' = a^*$, or $a_k' \neq 0$ and $a_{n+1}^* > a_{n+1}'$. Hence a^* is the element in $[b]$ with the largest a_{n+1}.

Solving the set of equations

$$b_i = \frac{a_i^* + a_{n+1}^*}{1+(n-1)a_{n+1}^*}, \quad i \neq n+1, \quad \text{and} \quad a_k^* = 0$$

for a_i^*, yields (58).

Since $a^* \in [b]$, then $a^*K \subseteq [b]$. Moreover, since, by (52), all elements in $[b]$ can be strictly ordered by the value of a_{n+1}, and $b = a^*e_0$ and a^*e_1 are the minimal and maximal elements of $[b]$, respectively, then $a^*K = [b]$. Since for any $a^* \neq a \in [b]$, $(ae_1)_{n+1} < (a^*e_1)_{n+1}$, a^* is the unique element in $S_0 \oplus S_1$ that generates $[b]$. □

Since equivalence classes inherit some of the properties of group-theoretic cosets, we will refer to them as *pseudocosets* and maintain the usual coset notation, a^*K. We call a^* the *pseudocoset generator* of a^*K, and the set of all a^* in S_1, S_1^*, the *pseudocoset generating set* for K in $S_0 \oplus S_1$.[4] It follows from theorem 2 that the direct sum

$$\bigoplus_{a^* \in S_1^*} a^*K \tag{59}$$

is the *pseudocoset decomposition* of K in $S_0 \oplus S_1$.[5]

Theorems 1 and 2 establish mathematical relationship between the restricted singleton DS set $S_0 \oplus S_1$ and the Bayes set S_0. An extension of this relationship to the unrestricted singleton set, S, will be given in section 5. Together, these results show that the operation on the DS mass, up to the uncertainty factor, is equivalent to the operation on Bayes probability. In

[4]Note, that if $a, a^* \in a^*K$ and $a \neq a^*$, then $aK \subset a^*K$, hence the standard notion of a coset, that is either disjoint or identical with any other coset, does not apply.

[5]One consequence of the fact that the factor set $S_0 \oplus S_1/K$ is a group, with relevance to DS theory, is that each element combined with another element of an appropriate pseudocoset yields an element in K.

particular, theorem 1 reveals that there is a natural correspondence between the elements of the Bayes set and the subsets of the DS set, while theorem 2 provides means of an explicit identification of the Bayes elements with the DS subsets. In practical terms, replacement of the DS operation (22) with the Bayes operation (35) results in computational complexity reduction, which becomes significant when the intermediate computations in a sequence of DS combinations are performed on unnormalized mass.

Example 3 Set $n = 2$. Then

$$K = \left\{ \left(\frac{1-u}{2}, \frac{1-u}{2}, u \right), \quad 0 \leq u \leq 1 \right\},$$

$$S_0 = \{(t, 1-t, 0), \quad 0 < t < 1\}$$

and, by (58),

$$a^* = \begin{cases} \left(0, \frac{1-2t}{1-t}, \frac{t}{1-t}\right), & t < \frac{1}{2}, \\ (0, 0, 1), & t = \frac{1}{2}, \\ \left(\frac{2t-1}{t}, 0, \frac{1-t}{t}\right), & t > \frac{1}{2}. \end{cases}$$

Fix $t = \frac{1}{3}$. Then $b = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$ and $a^* = \left(0, \frac{1}{2}, \frac{1}{2}\right)$. The pseudocoset a^*K contains, among others, the elements

$$\begin{aligned} a^* \circ e_0 &= \left(\frac{1}{3}, \frac{2}{3}, 0\right), \\ a^* \circ \left(\frac{3}{8}, \frac{3}{8}, \frac{1}{4}\right) &= \left(\frac{3}{13}, \frac{8}{13}, \frac{2}{13}\right), \\ a^* \circ \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) &= \left(\frac{1}{7}, \frac{4}{7}, \frac{2}{7}\right), \\ a^* \circ \left(\frac{1}{8}, \frac{1}{8}, \frac{3}{4}\right) &= \left(\frac{1}{15}, \frac{8}{15}, \frac{6}{15}\right), \quad \text{and} \\ a^* \circ e_1 &= \left(0, \frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

5 Computing in the unrestricted singleton DS set

In this section we extend the DS analysis to the set $S_0 \oplus S_1 \oplus S_0'$, the unrestricted singleton DS set.

Denote by $\mathbf{0}$ the $(n+1)$-tuple of all zeroes, set

$$B^0 = S_0 \oplus S_0' \oplus \mathbf{0}, \tag{60}$$

$$S^0 = S_1 \oplus B^0 = S \oplus \mathbf{0}, \tag{61}$$

and define the binary operation on S^0, $\circ_S$, by

$$a \circ_S b = \begin{cases} a \circ b, & < a, b > \neq 0, \\ \\ \mathbf{0}, & \text{otherwise.} \end{cases} \tag{62}$$

We have the following three results, which are slight generalizations of lemmas 2 and 4, and theorem 1, respectively.

Lemma 6 $(B^0, \circ_S)$ *is a monoid with the identity* e_0.

Lemma 7 $(S^0, \circ_S)$ *is a monoid with the identity* e_1.

Theorem 3 *The mapping*

$$\phi : a \mapsto a \circ_S e_0 \tag{63}$$

is a monoid homomorphism from S^0 *onto* B^0.

The homomorphism in (63) differs trivially from the homomorphism in section 4 in that it produces an additional collection of $|S_0' \oplus \mathbf{0}|$ degenerate pseudocosets, consisting of elements of $S_0' \oplus \mathbf{0}$. Theorem 2 applies to S^0 as well. Composition of a degenerate pseudocoset with any pseudocoset yields a degenerate pseudocoset.

Extending the DS analysis from S to S^0 allows one to gracefully address the case when the standard DS computation is not feasible. Apart from completing the algebra of singleton mass assignments, the extension provides a convenient mechanism for identifying incongruent bodies of evidence. Such evidence arises, for example, in diagnostic medicine, in identification of unknown diseases. Medical tests in such cases need to eliminate all known alternatives, rather than to point to the least unlikely possibility, to allow the hypothesis of a new alternative.

Apart from e_0, e_1 and $\mathbf{0}$, all elements in S_0', whose non-zero components are equal, are idempotents. These idempotents give rise to special subsets of B^0, called semilattices [4]. As identification of the DS process with a combination with an idempotent determines the outcome of evidence combination, semilattices play an important role in the investigation of asymptotic properties of the DS analysis. This topic will be addressed in a sequel.

The main focus of this paper was on the relationship between the DS set and the Bayes set. The last, well-known result shows that the Bayes set can be viewed, in a certain sense, as the limit of the DS analysis.

Theorem 4 *The DS combination of two elements in S, at least one of which is in* $S_1 \backslash \{e_1\}$*, reduces uncertainty.*

Proof

$$
\begin{aligned}
(a \circ b)_{n+1} &= \frac{a_{n+1}b_{n+1}}{1-\kappa} \\
&= \frac{a_{n+1}b_{n+1}}{1-(1-a_{n+1})(1-b_{n+1})+(a_1b_1+...+a_nb_n)} \\
&\leq \frac{a_{n+1}b_{n+1}}{1-(1-a_{n+1})(1-b_{n+1})} \\
&= \frac{a_{n+1}b_{n+1}}{a_{n+1}+b_{n+1}-a_{n+1}b_{n+1}} \\
&\leq min(a_{n+1}, b_{n+1}) \;\square
\end{aligned}
$$

We remark that there is a close connection between theorem 4 and the lack of inverses in $S_1 \backslash \{e_1\}$.

6 Conclusions

Developments of this paper reveal a close link between computations taking place in the singleton DS and Bayes sets, $S_0 \oplus S_1$ and S_0. This link is made precise by the homomorphism theorem. Since S_0 is an ideal of $S_0 \oplus S_1$[6], in cases when at least one of the elements to be combined is in S_0, computations in the two semigroups yield identical results. In the case

[6]It is worth pointing out the multiple roles S_0 plays in the restricted singleton DS analysis: (1) S_0 is the homomorphic image of $S_0 \oplus S_1$ under ϕ, (2) S_0 is an ideal of $S_0 \oplus S_1$, and (3) S_0 is the maximal subgroup of $S_0 \oplus S_1$. The last property permits viewing the DS analysis as a monoid extension of the Bayes analysis.

when both elements are in S_1, replacement of the computation in $S_0 \oplus S_1$ with a corresponding computation in S_0 results in loss of the uncertainty estimate, as homomorphism distributes uncertainty among all components of the composite mass. This might not be a significant drawback in situations when uncertainty is either very small or very large. In the former case the results of DS and Bayes evidence accumulations are similar. In the latter case (much less likely to occur as combination of evidence reduces uncertainty very quickly) the utility of results of either analysis is questionable. However, a loss of a potentially valuable information will occur when uncertainty assumes an intermediate value. This loss needs to be evaluated, for a given application, and compared with the gains brought about by the replacement of the DS approach with the Bayes approach, reduction of computational complexity of the combining operation and simplification of the analysis, before such replacement is made.

The above interpretation holds, if uncertainty assignments are understood as *absolute* error measures of mass evaluations. However, since DS mass evaluations are, in general, subjective, it might be reasonable to interpret uncertainty as a *relative* error, so that combination of two bodies of evidence gives larger weight to the body with lower uncertainty. Once the evidence is combined, the cumulative uncertainty ceases to have much interpretable value, as it represents only a part of the overall error (the other part being the error associated with subjective judgementis about mass assignment of individual bodies of evidence). This suggests that the distribution of uncertainty performed by the homomorphism does not need to result in a loss of information, even in the general case, and therefore the computation performed in S_0 might fully capture the complexity of the DS analysis in the larger semigroup.

One of the main findings of this work is the revelation of pseudocoset structure of the DS set. It is shown that elements of a fixed pseudocoset differ only by an affine scaling of its components, and that the maximal value of uncertainty depends on the smallest component of the homomorphic image of the pseudocoset. By the homomorphic relationship, combination of elements from two (not necessarily distinct) pseudocosets is equivalent to combination of the corresponding Bayes elements. This result decouples the choice among alternatives from the value of uncertainty. In effect, regardless of the viewpoint taken on the role of uncertainty, results of DS analysis can be directly interpreted in terms of Bayes probabilities.

While the focus of this paper was on the singleton case, the semigroup approach is applicable to the general case as well. The homomorphism extends trivially to the general case. The structure of equivalence classes, however, is more complex, and the pseudocoset generators are not unique. Despite the greater complexity of the general DS set, the fundamental, semigroup-theoretic relationship between the DS and Bayes analyses still holds and the interpretation of the role of uncertainty presented in this paper remains valid. Full account of the general formalism will be given in [7], together with a discussion of the role of semigroup ideals in the process of evidence elimination.

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