A QUASI-SYMMETRIC MODEL FOR THE TWO-WAY MIMO COMMUNICATION CHANNEL

Lang P. Withers, Jr.

The MITRE Corporation
7515 Colshire Drive, McLean, Virginia USA 22102-7508
email: lpwithers@mitre.org

ABSTRACT

It is well-known that the wireless MIMO fading channel at any moment in time and at any frequency is not exactly reciprocal or symmetric. This raises the question of how the forward and reverse channel response matrices are related. Clearly we do not want to forfeit all symmetry and merely consider them as two independent instances of a stochastic fading model. Here a more general relation, called **quasi-symmetry**, is proposed to relate the incoming and outgoing channel response matrices. We also find that quasi-symmetry is equivalent to simple relations between the complex symmetric singular value decompositions (SVDs) of the two channel matrices. Appropriate background is included on the two kinds of complex linear algebra that originate from the usual conjugate-symmetric scalar product, and the transpose-symmetric scalar product, respectively.

Index Terms— MIMO wireless channel, reciprocity, quasi-symmetry, complex symmetric matrices

1. INTRODUCTION

Transceivers with multiple antennas introduce spatial diversity to constructively combine multiple reflections of signal-carrying waves and to combat fading in the physical RF communication channel. Recent signal processing methods for MIMO transceivers, such as spacetime block coding and DBLAST, make use of current knowledge of the incoming channel at the receiver [1]. The best performance comes from methods that rely on knowledge of the outgoing channel at the transmitter as well [1][2]. Stochastic models for the one-way MIMO channel, such as those of Rayleigh, Rice, Clarke-Gans-Jakes, Suzuki, and others, have been studied in depth [3]. But to date little attention has been paid to the instantaneous two-way MIMO channel, in the sense of defining how the forward and reverse channels are related.

A first answer to the question of how they are related is to say that the channels are reciprocal or symmetric. This is true for electromagnetic waves sent and received between two antennas A and B in empty space, free of any scattering materials. We model the path response from A to B as an end-to-end gain coefficient h_{AB} times a complex phasor $\exp(i\omega_c\tau)$ for carrier frequency ω_c and path delay τ between a pair of antennas. The path delay, path loss, and gain at each antenna are the same in either direction, so $h_{BA} = h_{AB}$. This is a **reciprocal** or symmetric relation for the outgoing and incoming paths with respect to A. This reciprocal relation is still true when ideal reflectors are put into the space, which may create multiple paths between A and B or block the direct line-of-sight path, causing interference and fading. The reflectors and A and B can be moving, causing Doppler shifts, but reciprocity holds true.

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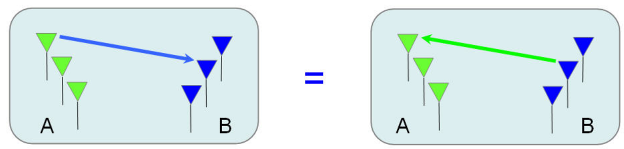

Fig. 1. Symmetric or reciprocal channel: $H_{BA} = (H_{AB})^T$.

Next consider a two-way communication channel between two MIMO transceivers A and B, with m_A and m_B antenna elements, resp. This MIMO channel is said to be reciprocal when the incoming and outgoing channel matrices H_{BA} and H_{AB} are such that

$$\boxed{H_{BA} = H_{AB}^T} \tag{1}$$

at each moment in time and for each narrow subband of the frequency band being used. This is true since for each pair of antennas, antenna j at A, antenna k at B, the response entry h_{kj}^{AB} in the matrix H_{AB} (to send from j at A to k at B) equals the entry h_{jk}^{BA} in the matrix H_{BA} (to send from k at B to j at A). Figure 1 illustrates this equality.

However, this symmetry may be broken for actual physical channels. The measured fading coefficients for the incoming and outgoing wireless channel are usually not the same [1]. One reason for this is that the transceivers and antennas

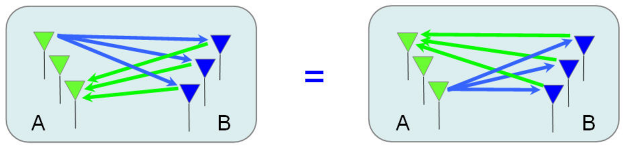

Fig. 2. Quasi-symmetric channel: $H_{BA}H_{AB} = (H_{BA}H_{AB})^T$ (pictured for elements 1 and 3 of array A) and $H_{AB}H_{BA} = (H_{AB}H_{BA})^T$.

may be different at A and B [4]. Physical effects that produce channel asymmetry or non-reciprocity include carrier wave diffraction around barriers, doppler shifts for scatterers with frequency-dependent absorption, and path variation due to motion of A and B or the media during long propagation delays.

As a more realistic property of the two-way channel, we propose a kind of roundtrip symmetry. An asymmetric channel is said to be **quasi-symmetric** when the roundtrip channel matrices are both transpose symmetric:

$$\boxed{(H_{BA}H_{AB})^T = H_{BA}H_{AB}} \quad (2)$$

$$\boxed{(H_{AB}H_{BA})^T = H_{AB}H_{BA}.} \quad (3)$$

It is clear that any reciprocal channel is quasi-reciprocal. Figure 2 illustrates the meaning of (2) in terms of the roundtrip paths between any two antennas k, ℓ of A: the linear superposition of roundtrip channel coefficients for all paths from k to ℓ equals that for all paths from ℓ to k. (The coefficient h for each one-way path, e.g. from an antenna of A to an antenna of B, represents the total radiation that passes between the two antennas, e.g. actual multipath.) Let $G_{ABA} = (H_{BA}H_{AB}) = [g_{k\ell}]$. Then (2) means that $g_{k\ell} = g_{\ell k}$, or $\sum_j h^{BA}_{kj}h^{AB}_{j\ell} = \sum_j h^{BA}_{\ell j}h^{AB}_{jk}$, as shown in Figure 2 for antennas $k, \ell = 1, 3$. The meaning of (3) is of course similar, but for roundtrip paths that begin and end on any two of B's antennas.

1.1. Example of a quasi-symmetric channel

We can show that a MIMO system with asymmetric transceivers has a quasi-symmetric two-way channel. Let us factor the forward channel matrix as $H_{AB} = R_B P_{AB} T_A$, where the complex diagonal matrices $T_A = t_A I_{m_A}$ and $R_B = r_B I_{m_B}$ represent the transmit and receive chains up to the antennas, resp. This assumes that the synchronized transmit chains and receive chains have uniform values within the array on each side A or B, but different values for the two arrays: $t_A \neq t_B$, $r_A \neq r_B$. (If the chain values within an array are nonuniform, digital compensation may be possible since it is a local effect.) Here P_{AB} represents the wireless propagation channel, including antenna gains for all paths. We assume it is reciprocal: $P_{AB}^T = P_{BA}$. We model the reverse channel matrix similarly as $H_{BA} = R_A P_{BA} T_B$.

Then the two-way channel is not reciprocal ($H_{BA} \neq H_{AB}^T$), but it is quasi-symmetric. For we have $(H_{BA}H_{AB}) = R_A\left[P_{BA}T_B R_B P_{AB}\right]T_A$. The factor $Q = [P_{BA}T_B R_B P_{AB}]$ is symmetric since $T_B R_B = R_B T_B$; that is, $Q = [q_{jk}] = Q^T = [q_{kj}]$. Then $R_A Q T_A = [r^A_{jj} q_{jk} t^A_{kk}]$ is also symmetric, since $r^A_{jj}t^A_{kk} = r^A_{kk}t^A_{jj} = r_A t_A$, for uniform transceiver chains on side A. The same reasoning shows that $(H_{AB}H_{BA})$ is also symmetric. Therefore, by our twofold definition (2)(3), H_{AB} and H_{BA} have a quasi-symmetric relation.

The outline for the rest of this paper is as follows. Section 2 gives an overview of the two kinds of linear algebra that we will be using for complex vectors and matrices. In Section 3, we present an equivalent version of quasi-symmetry in terms of SVDs. In Section 4, we discuss how to apply quasi-symmetry in SVD form to two-way channel estimation. We conclude in Section 5.

2. TWO TYPES OF LINEAR ALGEBRA

There are two ways to transpose a complex matrix $A = [a_{jk}] \in \mathbb{C}^{m\times n}$: with or without complex conjugation of its elements. Without, its transpose is $A^T = [a_{kj}]$. With, its transpose is $A^H = \bar{A}^T = [\bar{a}_{kj}]$. What is perhaps surprising is that both transposes are found together in complex signal processing.

The n-dimensional linear space $\mathbb{C}^n$of complex vectors has *two* scalar products. With the inner product $\langle \mathbf{x}, \mathbf{y}\rangle_H = \mathbf{x}^H\mathbf{y} = \sum_i \bar{x}_i y_i$, $\mathbb{C}^n$ is called a Hilbert space. (For A not positive definite and $A^H = A$, $\langle \mathbf{x}, A\mathbf{y}\rangle_H$ is an indefinite scalar product [5].) The other scalar product is $\langle \mathbf{x}, \mathbf{y}\rangle_T = \mathbf{x}^T\mathbf{y} = \sum_i x_i y_i$. Of course they are the same for real vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Both scalar products are non-degenerate, in the sense that if for all $\mathbf{y} \in \mathbb{C}^n$, $\langle \mathbf{x}, \mathbf{y}\rangle = 0$, then $\mathbf{x} = \mathbf{0}$. Both induce a vector 2-norm in the usual way: we define $|\mathbf{x}|^2_H = \langle \mathbf{x}, \mathbf{x}\rangle_H$, $|\mathbf{x}|^2_T = \langle \mathbf{x}, \mathbf{x}\rangle_T$. But only the first norm squared is positive and 0 only when $\mathbf{x} = \mathbf{0}$. The second can be negative, and can be 0 for non-zero vectors called null vectors (e.g., $|(1, i)|^2_T = 1^2 + i^2 = 0$) [6]. Only the inner product obeys the Schwarz inequality. Linear independence and rank do not depend on which scalar product we use, but orthogonality does.

To provide perspective (and remind us that sometimes we must venture outside of Hilbert space), consider some examples involving both types of transpose:

Example 1: The discrete Fourier transform (DFT) matrix is symmetric and unitary: $F^T = F$ and $F^H F = I$, where $F = [\omega_n^{jk}]$, $\omega_n = e^{i2\pi/n}$. (Also $F^4 = I$.)

Example 2: Given only the complex inner product $\langle \mathbf{x}, \mathbf{y}\rangle_H$, we can not recover the four real inner products of the real and imaginary parts of the two complex vectors. Write $\mathbf{x} = \mathbf{a} + i\mathbf{b}$, $\mathbf{y} = \mathbf{c} + i\mathbf{d}$, for $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$. If we have *both* scalar products, $\langle \mathbf{x}, \mathbf{y}\rangle_H = \mathbf{a}^T\mathbf{c} + \mathbf{b}^T\mathbf{d} + i\left(\mathbf{b}^T\mathbf{c} + \mathbf{a}^T\mathbf{d}\right)$, and $\langle \mathbf{x}, \mathbf{y}\rangle_T = \mathbf{a}^T\mathbf{c} - \mathbf{b}^T\mathbf{d} + i\left(-\mathbf{b}^T\mathbf{c} + \mathbf{a}^T\mathbf{d}\right)$, it is clear we can recover the individual real inner products $\langle \mathbf{a}, \mathbf{c}\rangle$, $\langle \mathbf{a}, \mathbf{d}\rangle$,

$\langle \mathbf{b}, \mathbf{c} \rangle$, $\langle \mathbf{b}, \mathbf{d} \rangle$, by taking sums and differences. This is essentially why, to specify all auto-correlation statistics of a complex stationary process (e.g., circular complex Gaussian noise) we must use both scalar products [7]. The same is true for complex cross-correlations, and higher-order correlations.

Example 3: Suppose $A = A^T$, with SVD $A = \sum \sigma \mathbf{u}\mathbf{v}^H$. Since the SVD is unique up to a phasor multiple ρ of each pair $\mathbf{u}$ and $\mathbf{v}$, for $|\rho| = 1$, $A = A^T$ implies $\mathbf{u} = \rho\bar{\mathbf{v}}$. Then $\rho = \langle \mathbf{u}, \mathbf{u} \rangle_T / \langle \mathbf{u}, \mathbf{v} \rangle_H$. To balance the SVD, replace $\mathbf{u}$ by $\mathbf{w} = \rho^{-\frac{1}{2}}\mathbf{u}$. Then $\langle \mathbf{w}, \mathbf{w}' \rangle_H = \delta_{\mathbf{w}\mathbf{w}'}$, and $A = \sum \sigma \mathbf{w}\mathbf{w}^T$. This is the unique Takagi form of the SVD.

Example 4: The real and imaginary parts of a unitary matrix U are quasi-symmetric. Let $U = A + iB$, with Re$(U) = A$, Im$(U) = B$, so that $U^H U = UU^H = I$. Then
$(A - iB)^T(A + iB) = A^T A + B^T B + i(A^T B - B^T A) = I$,
$(A - iB)(A + iB)^T = AA^T + BB^T + i(AB^T - BA^T) = I$.
Equivalently, $A^T A = I - B^T B$, $AA^T = I - BB^T$, and $A^T B = B^T A$, $AB^T = BA^T$. From the two imaginary conditions, A^T and B are quasi-symmetric.

The two scalar products generate two parallel versions of complex linear algebra. An orthonormal basis exists for $\mathbb{C}^n$ with either scalar product, and any vector can be expanded in terms of the basis [6]. The decompositions for $\langle \cdot, \cdot \rangle_H$ (Gram-Schmidt, EVD, SVD, polar) have analogs for $\langle \cdot, \cdot \rangle_T$, even with null vectors present [8][9][10]. It is always true that rank$(A^H A)$ = rank(A), but due to null column vectors of A, rank$(A^T A)$ may be less than rank(A) for a complex matrix A [11]. For simplicity and brevity, in this paper we assume all matrices A are well-behaved, i.e., have rank$(A^T A)$ = rank(A).

2.1. Two SVDs of a Complex Matrix

We give a short account of two SVDs (HSVD and TSVD) that result from the two scalar products. We will need the TSVD in the next section, to characterize quasi-symmetry. (Two hybrid SVDs, for H- and T-orthogonal left and right vectors, also exist [14], but are not needed here.) The theorems and proofs in the sequel are presented in a parallel manner: we let $\langle \cdot, \cdot \rangle$ stand for $\langle \cdot, \cdot \rangle_H$ or $\langle \cdot, \cdot \rangle_T$; for a matrix A, A^* stands for A^H or A^T; for a scalar α, α^* stands for its conjugate $\bar{\alpha}$ or α; resp. We say A is symmetric if $A^* = A$, and define A and B to be quasi-symmetric if AB and BA are symmetric.

A basic principle is that a matrix with either kind of symmetry has eigenvectors that are orthogonal in kind. In the case $A^H = A$, the eigenvalues are easily shown to be real. In the case $A^T = A$, eigenvectors can be null vectors.

Lemma 1. Let A be an $n \times n$ complex matrix. Suppose $A^* = A$. Then eigenvectors of distinct eigenvalues of A are orthogonal.

Proof: Suppose $A\mathbf{v} = \alpha\mathbf{v}$, and $A\mathbf{v}' = \beta\mathbf{v}'$, for $\alpha \neq \beta$. Then $\langle \mathbf{v}', \mathbf{A}\mathbf{v} \rangle = \alpha\langle \mathbf{v}', \mathbf{v} \rangle$, and $\langle \mathbf{v}', \mathbf{A}\mathbf{v} \rangle = \langle \mathbf{A}^*\mathbf{v}', \mathbf{v} \rangle = \langle \mathbf{A}\mathbf{v}', \mathbf{v} \rangle = \beta^*\langle \mathbf{v}', \mathbf{v} \rangle = \beta\langle \mathbf{v}', \mathbf{v} \rangle$. Since $\alpha \neq \beta$, and there are no zero-divisors in $\mathbb{C}$, it follows that $\langle \mathbf{v}', \mathbf{v} \rangle = 0$. □

The two kinds of SVD of a matrix A, denoted by TSVD and HSVD, can both be derived from Lemma 1. In a nutshell, since $A^* A$ is symmetric, its orthogonal eigenvectors $\mathbf{v}$ are taken as right singular vectors, and orthogonal vectors $\mathbf{u} = A\mathbf{v}/\sqrt{\alpha}$ as left singular vectors. Then the SVD is $A = \sum \sqrt{\alpha}\mathbf{u}\mathbf{v}^*$ [12][11]. For a full account of the TSVD, see [8].

3. QUASI-SYMMETRY AS AN SVD RELATION

We are now prepared to characterize quasi-symmetric matrices, in terms of their SVDs.

There are two special cases when A and B are quasi-symmetric. First, if $B = A^*$, it is clear that AB and BA are symmetric. Second, if A and B themselves are symmetric, quasi-symmetry immediately reduces to usual commutativity of A and B.

As motivation, we begin with a well-known way (key to quantum mechanics) to characterize commuting symmetric matrices in terms of their eigenvalue decompositions (EVDs).

Theorem 2. Let $A, B \in \mathbb{C}^{n \times n}$ each be symmetric. Then $AB = BA$ if and only if A and B have the same eigenvectors (not the same eigenvalues).

Proof: Suppose $A = \sum \alpha\mathbf{v}\mathbf{v}^*$, $B = \sum \beta\mathbf{v}\mathbf{v}^*$. Then it is clear that $AB = BA$. Conversely, suppose $AB = BA$. Assume that A and B each have a distinct eigenvalue for every eigenvector. Let A have EVD $A = \sum \alpha\mathbf{v}\mathbf{v}^*$. Then for any eigenvector $\mathbf{v}_0$ of A,

$$A(B\mathbf{v}_0) = BA\mathbf{v}_0 = \alpha_0(B\mathbf{v}_0), \tag{4}$$

Since $B\mathbf{v}_0$ is an eigenvector of A with the same eigenvalue α_0 as $\mathbf{v}_0$, the two vectors must be equal, up to some scalar β_0: $B\mathbf{v}_0 = \beta_0\mathbf{v}_0$. Thus $\mathbf{v}_0$ is also an eigenvector of B, but with eigenvalue β_0. The proof when the eigenvalues of A or B are not distinct is too long to include here. □

We can generalize Theorem 2 and its proof for quasi-symmetric rectangular matrices A and B, using their SVDs:

Theorem 3. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$, each with distinct singular values. Then AB and BA are symmetric if and only if the left and right singular vectors of A are the same as the right and left singular vectors of B, resp. (the singular values of A and B need not be the same).

Proof: Suppose $A = \sum \alpha\mathbf{u}\mathbf{v}^*$, $B = \sum \beta\mathbf{v}\mathbf{u}^*$. Then it is clear that $AB = \sum \alpha\beta\mathbf{u}\mathbf{u}^*$ and $BA = \sum \alpha\beta\mathbf{v}\mathbf{v}^*$, so that AB and BA are symmetric. Conversely, suppose $AB = \sum \rho\mathbf{u}\mathbf{u}^*$ and $BA = \sum \sigma\mathbf{v}\mathbf{v}^*$. Assume that A and B each have a distinct (unique) singular value for every pair of singular vectors. Then for any eigenvector $\mathbf{u}_0$ of AB,

$$BA(B\mathbf{u}_0) = B(AB\mathbf{u}_0) = \rho_0(B\mathbf{u}_0). \tag{5}$$

Since $B\mathbf{u}_0$ is an eigenvector of BA, it must equal one of its eigenvectors, $\mathbf{v}_0$, up to some scalar β_0: $B\mathbf{u}_0 = \beta_0\mathbf{v}_0$, so that $\mathbf{u}_0$, $\mathbf{v}_0$ are a unique pair of right and left singular vectors of B. Since $\langle \mathbf{v}_0, \mathbf{v}_0 \rangle = 1$, $\beta_0 = \langle \mathbf{v}_0, B\mathbf{u}_0 \rangle$. Similarly, for any eigenvector $\mathbf{v}_0$ of BA, we find $A\mathbf{v}_0 = \alpha_0\mathbf{u}_0$,where $\alpha_0 = \langle \mathbf{u}_0, A\mathbf{v}_0 \rangle$. Thus $\mathbf{v}_0$, $\mathbf{u}_0$ are a unique pair of right and left singular vectors of A. Our space is again too short to prove the case for non-distinct eigenvalues of AB or BA. □

Theorem 3 was first noted by Eckart and Young, in their paper introducing the general HSVD [13] [11].

Corollary 4. A unitary matrix U has TSVD of the form $U = \sum \rho \mathbf{x}\mathbf{y}^T$ for real orthonormal vectors $\mathbf{x}$ and $\mathbf{y}$, and $|\rho| = 1$.

Proof: Let $U = A + iB$ as in Example 4, where we saw that A^T and B are quasi-symmetric. Their SVDs are real, and by Theorem 3 are related as $A = \sum \alpha \mathbf{x}\mathbf{y}^T$ and $B = \sum \beta \mathbf{x}\mathbf{y}^T$. Therefore, $U = \sum(\alpha + i\beta)\mathbf{x}\mathbf{y}^T$ for real orthonormal vectors $\mathbf{x}$ and $\mathbf{y}$. Substituting this sum into $U^H U = I$, on the diagonal we have $|\alpha + i\beta|^2 = 1$. This is the TSVD of U. □

4. THE QUASI-SYMMETRIC CHANNEL

If a wireless channel is quasi-symmetric in the sense of (2)(3), then by Theorem 3, the TSVDs of the forward and reverse channel matrices have the form

$$\boxed{H_{AB} = \textstyle\sum_k \sigma_k \mathbf{u}_k \mathbf{v}_k^T} \tag{6}$$

$$\boxed{H_{BA} = \textstyle\sum_k \rho_k \mathbf{v}_k \mathbf{u}_k^T.} \tag{7}$$

Quasi-symmetry could be used to constrain H_{AB} and H_{BA} when estimating the channel in both directions from training data exchanged between stations A and B. For example, quasi-symmetry might be used in the echo-MIMO method, which estimates H_{AB} from $(H_{BA}H_{AB})$ and H_{BA} [2].

The usual HSVD of the channel matrix H_{AB} also plays a role, because optimum matched beamforming at arrays A and B depends on it. For example, by the Schwarz inequality, the principal left and right singular vectors $\mathbf{y}$ and $\mathbf{x}$ of the HSVD of H_{AB} are the beamformers for A and B, resp., that give maximum link gain $\gamma = \langle \mathbf{y},\ H_{AB}\mathbf{x} \rangle_H$ to send a single signal stream [1]. The question of how to recover the HSVD directly from the TSVD of a matrix such as H_{AB} remains open.

5. CONCLUSION AND FUTURE WORK

A new quasi-symmetric model to relate the incoming and outgoing channel response at one end of a MIMO wireless communication link has been proposed. The relation is more general and flexible than simple reciprocity, but preserves round-trip symmetry on both sides of the link. As background, we briefly considered two parallel types of linear algebra that occur in processing complex signals, based on taking transposes with and without conjugation. Then we showed that quasi-symmetry has an equivalent expression in terms of the TSVDs of the forward and reverse channel matrices.

We demonstrated that quasi-symmetry holds for a two-way MIMO channel consisting of a symmetric propagation channel with asymmetric transmit and receive chains on the two ends of the link. Experimental validation of the quasi-symmetric relation for real propagation channels and transceiver arrays remains to be carried out in field tests in various urban and rural environments. The TSVD form of quasi-symmetry should be a useful way to test forward and reverse channel data from measurements for quasi-symmetry.

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